Mathematics++

Practicals 5 – Functional analysis

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All the vector spaces (also called linear spaces) are over the field $\mathbb{R}$.

Definition: Let E be a normed linear space. A **closed hyperplane** is every set of the form $H = \{x \in E : f(x) = \alpha\}$ where $f \in E^*$, $f \neq 0$ and $\alpha \in \mathbb{R}$. (This is the same as translations of maximal proper subspaces).

Spaces and norms:

- $\mathcal{C}([0,1])$: continuous functions $[0,1] \to \mathbb{R}$ with norm $\|f\| = \max\{|f(t)| : t \in [0,1]\}$.
- c: convergent sequences with norm $\|x_n\| = \sup\{|x_n| : n \in \mathbb{N}\}$.
- c_0: sequences convergent to 0, subspace of c.
- l^∞: bounded sequences, same norm as in c.
- $\mathcal{L}^p$: measurable functions on X with norm $\|f\|_p = \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p}$. This is not a norm, functions that are zero almost everywhere have norm zero.
- L^p: $\mathcal{L}^p$ modulo functions that are zero almost everywhere.

1. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x - y|$ but f is not a contraction.
2. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x - y|$ but f has no fixed point.
3. Show that every subspace of a normed linear space of finite dimension is closed. Find a counterexample for a space of infinite dimension.
4. Show that complement of every proper subspace of a normed linear space is dense.
5. Show that unit ball in a Hilbert space of inifinite dimension is not compact.
6. Prove Mazur theorem: Let C be an open convex subset of a normed linear space E and $z \in E \setminus C$. Then there exists a closed hyperplane $H \subset E$ such that $z \in H$ and $H \cap C = \emptyset$.
7. Decide whether following functionals on a normed linear space X are linear and continuous. If so, determine their norm.
   (a) $F : (x_n)_{i \in \mathbb{Z}^+} \mapsto \sum_{i=1}^{\infty} \frac{x_i}{i^2}$, $X = c_0$
   (b) $F : f \mapsto \int_0^1 t f(t) \, \mathrm{d}t$, $X = L^p([0,1])$
   (c) $F : f \mapsto \lim_{n \to \infty} \int_0^1 f(t^n) \, \mathrm{d}t$, $X = \mathcal{C}([0,1])$
