Mathematics++

Practicals 4 – Measure concentration

April 25th, 2022

Theorem 1 (Measure concentration theorem for spheres)**.** *For all $t \geq 0$ and $A \subseteq S^{n-1}$ with $\mu(A) \geq 1/2$ we have that $1 - \mu(A_t) \leq 2e^{-t^2 n/2}$.*

Theorem 2 (Concentration of Lipschitz functions - Levy's lemma)**.** *Let $f\colon S^{n-1} \to \mathbb{R}$ be 1-Lipschitz, then there exists $m \in \mathbb{R}$ such that for all $t \in (0,1]$ we have that*

$$\mu\{x \in S^{n-1}\colon f(x) \notin [m-t, m+t]\} \leq 4e^{-t^2 n/2}.$$

1. Given $t \in [-1,1]$, let $C(t)$ be the spherical cap of height $1-t$. That is,

   $$C(t) := \left\{x \in S^{n-1} : x_1 \geq t\right\}.$$

   For $t \in [0,1]$ prove that $\mu(C(t)) \leq e^{-t^2 n/2}$, where $\mu(C(t)) := \lambda(\tilde{C}(t))/\lambda(B^n)$ and $\tilde{C}(t)$ is the cone over $C(t)$ with apex in the origin.
   Hint: Show that $\tilde{C}(t)$ is a subset of a suitable ball. (For small t try to get radius $\sqrt{1-t^2}$. For larger t use another ball.)

2. Deduce the measure concentration for the dot product of two random unit vectors in $\mathbb{R}^n$.

3. Estimate the probability that a random line passing through origin intersects the unit ball centered at $(2,0,0,\ldots,0)$.

4. Using Lévy's lemma, show that the expected value $\int_{S^{n-1}} f(x)\,\mathrm{d}\mu(x)$ of a 1-Lipschitz function $f\colon S^{n-1} \to \mathbb{R}$ has distance at most $O(1/\sqrt{n})$ from the median.