Mathematics++

Practicals 1 – Measure and σ-algebras

February 28, 2022

Definition: A set is *dense*, if it has non-empty intersection with every non-empty open set.

Definition: Given a set system $\mathcal{H}$, the minimal σ-algebra *generated* by $\mathcal{H}$ is defined as a minimal σ-algebra containing $\mathcal{H}$. A *Borel set* is an element of σ algebra generated by all open balls (intervals of finite length in $\mathbb{R}^1$).

Definition: Let $(X, \mathcal{S}_X, \mu_X)$ and $(Y, \mathcal{S}_Y, \mu_Y)$ be measurable spaces. A function $f\colon X \to Y$ is *measurable* if $f^{-1}(S) \in \mathcal{S}_X$ for all $S \in \mathcal{S}_Y$.

Definition: A real function is *measurable* if it is measurable as above with respect to Lebesgue measurable sets in the preimage and the Borel sets in the image.

1. Provide an example of a subset of $\mathbb{R}$ which is both open and closed. Provide an example of a subset which is neither open nor closed.
2. Provide an example of a dense subset of $\mathbb{R}$.
3. Show that every open subset of $\mathbb{R}$ is a union of countably many intervals. [*]
4. Show that the intersection of an arbitrary collection of σ-algebras (over the same ground set) is again a σ-algebra.
5. Show that the complement of a Lebesgue measurable set is Lebesgue measurable.
6. Prove that the interval $(0, +\infty)$ is Lebesgue measurable.
7. Let $(X, \mathcal{S}, \mu)$ be a measurable space, where $\mathcal{S}$ is a finite σ-algebra. Describe measurable functions $X \to \mathbb{R}$.
8. Show that the (Lebesgue) measure of $\mathbb{Q}$ equals 0. [*]
9. Construct a compact subset of $\mathbb{R}$ of positive measure such that the complement of this set is dense. [*]
10. Decide which of the following subsets of $\mathbb{R}$ are Borel sets: $\mathbb{N}$, $\mathbb{R} \setminus \mathbb{Q}$, closed intervals.
11. Show that a real function f is measurable if and only if the set $\{x : f(x) < \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$. [**]
12. Prove that for measurable sets A, B and a measure μ, we have $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.
