

Mathematics++

Problem set 1 – Measure and σ -algebras

hints after **March 14, 2022**, solutions due **March 21, 2022** before the lecture

Send your solutions to `chmel@kam.mff.cuni.cz`

Definition: Let $x \in \mathbb{R}$ and $E \subset \mathbb{R}$ be a measurable set. We define the **density** of E at x as the limit

$$d_E(x) := \lim_{\delta \rightarrow 0} \frac{\lambda((x - \delta, x + \delta) \cap E)}{2\delta}$$

if such limit exists.

1. Prove or disprove that there exists an infinite σ -algebra with countably many elements only. [5*]
2. Show that the union of two Lebesgue measurable sets is Lebesgue measurable. [2]
3. Let \mathcal{S} be a σ -algebra and let $\mu : \mathcal{S} \rightarrow \overline{\mathbb{R}}_0^+$ be additive (i.e., for any disjoint $A, B \in \mathcal{S}$, $\mu(A) + \mu(B) = \mu(A \cup B)$). Show that the following are equivalent:
 - μ is lower semi-continuous, that is, for any $B_1 \subset B_2 \subset \dots \in \mathcal{S}$, we have $\mu(\bigcup_{n=1}^{\infty} B_n) = \lim_{n \rightarrow \infty} \mu(B_n)$,
 - μ is countably additive, that is, for any disjoint $A_1, A_2, \dots \in \mathcal{S}$, we have $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ [2]
4. Let (X, \mathcal{S}, μ) be a measurable space and $\{A_i\}_{i=0}^{\infty}$ be a sequence of measurable sets such that $A_{i+1} \subseteq A_i$ for every i . Assuming $\mu(A_0) < \infty$, show that

$$\lim_{i \rightarrow \infty} \mu(A_i) = \mu\left(\bigcap_{i=0}^{\infty} A_i\right).$$

In addition, show that the assumption above is necessary, that is, find a sequence as above which violates $\mu(A_0) < \infty$ as well as the conclusion. [4]

5. Prove that the set C defined below is measurable and determine its (Lebesgue) measure.

Let $\{\mathcal{K}_n\}$ be a sequence of finite collections of closed intervals defined inductively as

- $\mathcal{K}_0 = \{[0, 1]\}$, $\mathcal{K}_1 = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$,
- \mathcal{K}_n is obtained from \mathcal{K}_{n-1} by removing the open middle third of each of the intervals in \mathcal{K}_{n-1} .

Then, we set $K_n := \bigcup \mathcal{K}_n$ and $C := \bigcap_n K_n$. [4]

6. Show that every measurable set of finite measure can be approximated with arbitrary precision by a finite union of intervals; that is, $\forall E \subset \mathbb{R}$ of finite measure and $\forall \varepsilon > 0$ there is $A \subset \mathbb{R}$, which is a union of finitely many open intervals, such that $\lambda(E \Delta A) \leq \varepsilon$.

In addition, show that the assumption on finite measure is necessary; that is, find a measurable set of infinite measure which cannot be approximated by a finite union of intervals for some $\varepsilon > 0$. [4]

7. Prove that there is no measurable set $E \subseteq (0, 1)$ such that $\lambda(E) = 1/2$ and for every $x \in (0, 1)$ the density of E at x equals $1/2$. [6*]

Hint: You may want to use the approximation from the problem above.