Mathematics++

Problem set 5 – Functional analysis

Hints: **June 6th, 2022**. Deadline: **June 13th, 2022**. Send solutions to dbulavka+mpp@kam.mff.cuni.cz.

All the vector spaces (also called linear spaces) are over the field $\mathbb{R}$.

Definition A (topological) **dual** of a normed linear space E is the space of all bounded linear functions $E \to \mathbb{R}$ (so called functionals) together with the norm

$$\|F\| := \sup\{|Fx| : \|x\|_E \leq 1\}$$

and we denote the dual space E^*.

Theorem (Hahn-Banach): Let $f \in M^*$ be a contious linear function on M which is a subspace of a normed linear space E. Then there exists $F \in E^*$ such that $F = f$ on M and $\|F\|_E = \|f\|_M$.

Theorem (Fréchet-Riesz): Let L be a continous linear function on Hilbert space H. Then there exists excatly one $a \in H$ such that $L(x) = \langle x, a\rangle\ \forall x \in H$. Moreover $\|L\| = \|a\|$.

1. Let V be a linear space and $B \subseteq V$ its symmetric convex subset such that intersetion of B with every subspace of dimension 1 (which are exaclty the sets $\{\lambda x : \lambda \in \mathbb{R}\}$ for a fixed $x \neq 0$) is a closed interval of finite positive length. We define
$$\|x\|_B := \min\{k \geq 0 : x \in kB\}.$$
Prove that $\|.\|_B$ is a norm on V, and also that every norm on V can be defined with a suitable B. **[7]**

   Nápověda: Let $X \subseteq V$, show that X is convex if and only if $aX + bX = (a+b)X$ for all $a, b \geq 0$. For the second part consider $B = \{x \in V : \|x\| \leq 1\}$.

2. Given a linear space W with inner product show that:
$$\forall S \subseteq W : \overline{\langle S\rangle} = (S^\perp)^\perp$$ **[4]**

   Nápověda: First show it assuming that W is a Hilbert space. If W is not complete, consider its completion and note that $X^{\perp_W} = X^{\perp_{\overline{W}}} \cap W$.

3. Let X be the space of continous real functions on $[0,1]$. Show that no two norms $\|.\|_p$ for $p \in [1, \infty]$ are equivalent on this space. **[4]**

   Nápověda: Build a sequence of function which maximum value is constant, but which $\|f_n\|_p$ goes to 0.

4. Decide whether the following operators on a space X are linear and continuous. If so, calculate their norm ($\|L\| := \sup\{\|Lx\|_X : \|x\|_X \leq 1\}$): **[8]**
   (a) $Lf(t) := f(t^3)$, $X = \mathcal{C}([0,1])$
   (b) $Lf(t) := f(t^3)$, $X = L^2([0,1])$
   (c) $L(x_n)_{i\in\mathbb{N}} := (0, x_0, x_1, x_2, \ldots)$, $X = \ell^1$

(d) $L\left(x_n\right)_{i\in\mathbb{N}} := (x_1, x_2, x_3, \ldots)$, $X = \ell^1$

Nápověda:

5. Prove the following geometric version of Hahn-Banach theorem: Let A and B be non-empty open disjoint convex subsets of a normed linear space E. Then there exists (nonzero) $f \in E^*$ and $\alpha \in \mathbb{R}$ such that $A \subset \{x \in E : f(x) > \alpha\}$ and $B \subset \{x \in E : f(x) < \alpha\}$. **[4]**

   Nápověda: First show the following: take an open convex set $C \subseteq E$ with $0 \in C$. For $x \in E$ set $p(x) = \inf\{\alpha > 0 \colon x \in \alpha C\}$. Show that (1) p is in the conditions of the Hahn-Banach lemma, (2) there exists a constant M such that $p(x) \leq M$, and (3) $C = \{x \in E \colon p(x) \leq 1\}$.

   Next, show that for $C \subseteq E$ non-empty open convex set, let $x_0 \in E \setminus C$. Then, there exists $f \in E^*$ such that $f(x) < f(x_0)$ for all $x \in C$.

   Finally, to prove the exercise consider $C = A - B$ and show that it fits the above hypothesis.

6. Show that an orthonormal basis of a Hilbert space of infinite dimension cannot also be its algebraic (also known as Hamel) basis. Moreover show that algebraic basis of a Hilbert space of inifinite dimension is always uncountable. **[4]**

   Nápověda: For the first part, build an element that uses all the elements of an orthonormal basis. For the second part use that intersection of countable many open dense sets is dense.