Mathematics++

Problem set 5 – Functional analysis

Hints: **June 6th, 2022**. Deadline: **June 13th, 2022**. Send solutions to dbulavka+mpp@kam.mff.cuni.cz.

All the vector spaces (also called linear spaces) are over the field $\mathbb{R}$.

Definition A (topological) **dual** of a normed linear space E is the space of all bounded linear functions $E \to \mathbb{R}$ (so called functionals) together with the norm

$$\|F\| := \sup\{|Fx| : \|x\|_E \leq 1\}$$

and we denote the dual space E^*.

Theorem (Hahn-Banach): Let $f \in M^*$ be a contious linear function on M which is a subspace of a normed linear space E. Then there exists $F \in E^*$ such that $F = f$ on M and $\|F\|_E = \|f\|_M$.

Theorem (Fréchet-Riesz): Let L be a continous linear function on Hilbert space H. Then there exists excatly one $a \in H$ such that $L(x) = \langle x, a\rangle \ \forall x \in H$. Moreover $\|L\| = \|a\|$.

1. Let V be a linear space and $B \subseteq V$ its symmetric convex subset such that intersetion of B with every subspace of dimension 1 (which are exaclty the sets $\{\lambda x : \lambda \in \mathbb{R}\}$ for a fixed $x \neq 0$) is a closed interval of finite positive length. We define

$$\|x\|_B := \min\{k \geq 0 : x \in kB\}.$$

Prove that $\|.\|_B$ is a norm on V, and also that every norm on V can be defined with a suitable B. **[7]**

2. Given a linear space W with inner product show that:

$$\forall S \subseteq W : \overline{\langle S\rangle} = (S^{\perp})^{\perp}$$ **[4]**

3. Let X be the space of continous real functions on $[0,1]$. Show that no two norms $\|.\|_p$ for $p \in [1,\infty]$ are equivalent on this space. **[4]**

4. Decide whether the following operators on a space X are linear and continuous. If so, calculate their norm ($\|L\| := \sup\{\|Lx\|_X : \|x\|_X \leq 1\}$): **[8]**

   (a) $Lf(t) := f(t^3)$, $X = \mathcal{C}([0,1])$

   (b) $Lf(t) := f(t^3)$, $X = L^2([0,1])$

   (c) $L\,(x_n)_{i\in\mathbb{N}} := (0, x_0, x_1, x_2, \ldots)$, $X = \ell^1$

   (d) $L\,(x_n)_{i\in\mathbb{N}} := (x_1, x_2, x_3, \ldots)$, $X = \ell^1$

5. Prove the following geometric version of Hahn-Banach theorem: Let A and B be non-empty open disjoint convex subsets of a normed linear space E. Then there exists (nonzero) $f \in E^*$ and $\alpha \in \mathbb{R}$ such that $A \subset \{x \in E : f(x) > \alpha\}$ and $B \subset \{x \in E : f(x) < \alpha\}$. **[4]**

6. Show that an orthonormal basis of a Hilbert space of infinite dimension cannot also be its algebraic (also known as Hamel) basis. Moreover show that algebraic basis of a Hilbert space of inifinite dimension is always uncountable. **[4]**