Mathematics++

Problem set 4 – Measure Concentration

Hints: **May 16th, 2022**. Deadline: **May 23th, 2022**. Send solutions to dbulavka+mpp@kam.mff.cuni.cz.

1. Let B and B' be unit balls in $\mathbb{R}^n$ and let $d(n)$ be the distance between the centers of B and B' so that the probability that a uniformly randomly chosen point of B belongs to B' with probability 1 %. Find some explicit upper bound on $d(n)$ tending to 0 while n tends to infinity. **[3]**

   Nápověda: Estimate the intersection of the two spheres with a suitable smaller sphere.

2. Consider the unit cube $[0,1]^n$, n large, and take the hyperplane h given by the equation $x_1 + \cdots + x_n = n/2$. Estimate roughly the width of a parallel strip along h containing 99% of the volume of the cube.

   [3]

   Nápověda: Notice that the plane h is a translation of the plane $h' = \{x \in \mathbb{R}^n : \langle(1,\ldots,1), x\rangle = 0\}$, the distance of a point x to h' is given by $d(x, h') = \frac{1}{\sqrt{n}}|x_1 + \cdots + x_n|$. That is, the problem is to estimate the probability that the sum of n independent random variables deviate from their mean.

3. Let P be a convex polytope in $\mathbb{R}^n$ which is the intersection of N halfspaces. Let us assume that P contains the unit ball. Show that

   $$\lambda(P) \geq \left(C\frac{n}{\ln N}\right)^{n/2} \lambda(B^n)$$

   for some constant $C > 0$. *Hint:* Consider a sphere such that the complement of P covers exactly one half of it's surface area. **[5*]**

   Nápověda: Use the hint. The covered part of the constructed sphere is formed by the union of spherical caps. How many are there and how big are they (relative to the surface of the whole sphere)?

4. A set $N \subseteq S^{n-1}$ is called ϵ-dense if every $x \in S^{n-1}$ is at distance at most ϵ from some $y \in N$.

   (a) Show that any 1-dense set in S^{n-1} has at least $\frac{1}{2}e^{n/8}$ points. **[2]**

   Nápověda: Consider a spherical cap centered around each point of the 1-dense set.

   (b) Show that for every $\delta \in (0,1]$ there is δ-dense set in S^{n-1} of size at most $(4/\delta)^n$. **[2]**

   Nápověda: Consider an inclusion-maximal set in which every two points have distance greater than δ, and Euclidean balls of radius $\delta/2$ around each of its points.

5. Construct a continuous function $f : S^{n-1} \to [0,1]$, which is not concentrated. That is, $\forall x \in [0,1]$, the set $f^{-1}([x - \frac{1}{3}, x + \frac{1}{3}])$ has measure at most $\frac{2}{3}$. **[3]**

   Nápověda: Start with a very simple function attaining only values 0 and 1.

6. Derive the measure concentration theorem for spheres from Levy's lemma, perhaps with a worse constant in front of $e^{-t^2 n/2}$. **[2]**

   Nápověda: For a fix subset $A \subseteq S^{n-1}$, the function $f : S^{n-1} \to \mathbb{R}$ given by $f(x) = dist(x, A)$ is 1-Lipschitz.