Mathematics++

Problem set 4 – Measure Concentration

Hints: **May 16th, 2022**. Deadline: **May 23th, 2022**. Send solutions to dbulavka+mpp@kam.mff.cuni.cz.

1. Let B and B' be unit balls in $\mathbb{R}^n$ and let $d(n)$ be the distance between the centers of B and B' so that the probability that a uniformly randomly chosen point of B belongs to B' with probability 1 %. Find some explicit upper bound on $d(n)$ tending to 0 while n tends to infinity. **[3]**

2. Consider the unit cube $[0,1]^n$, n large, and take the hyperplane h given by the equation $x_1 + \cdots + x_n = n/2$. Estimate roughly the width of a parallel strip along h containing 99% of the volume of the cube. **[3]**

3. Let P be a convex polytope in $\mathbb{R}^n$ which is the intersection of N halfspaces. Let us assume that P contains the unit ball. Show that

$$\lambda(P) \geq \left(C\frac{n}{\ln N}\right)^{n/2} \lambda(B^n)$$

for some constant $C > 0$. *Hint:* Consider a sphere such that the complement of P covers exactly one half of it's surface area. **[5*]**

4. A set $N \subseteq S^{n-1}$ is called ϵ-dense if every $x \in S^{n-1}$ is at distance at most ϵ from some $y \in N$.

   (a) Show that any 1-dense set in S^{n-1} has at least $\frac{1}{2}e^{n/8}$ points. **[2]**

   (b) Show that for every $\delta \in (0,1]$ there is δ-dense set in S^{n-1} of size at most $(4/\delta)^n$. **[2]**

5. Construct a continuous function $f : S^{n-1} \to [0,1]$, which is not concentrated. That is, $\forall x \in [0,1]$, the set $f^{-1}([x-\frac{1}{3}, x+\frac{1}{3}])$ has measure at most $\frac{2}{3}$. **[3]**

6. Derive the measure concentration theorem for spheres from Levy's lemma, perhaps with a worse constant in front of $e^{-t^2 n/2}$. **[2]**