Mathematics++

Problem set 3 – Convex geometry

Hints: **April 25th, 2022**. Deadline: **May 2nd, 2022**. Send solutions to dbulavka+mpp@kam.mff.cuni.cz.

Prékopa-Leindler inequality: Let $t \in (0,1)$ and let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}$ be measurable, non-negative, bounded functions with a finite integral over $\mathbb{R}^n$. Suppose that $h((1-t)x+ty) \geq f(x)^{1-t} g(y)^t$ for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-t} \left(\int_{\mathbb{R}^n} g\right)^{t}.$$

Log-concavity: A non-negative function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called log-concave if it satisfies

$$f((1-t)x+ty) \geq f(x)^{1-t} f(y)^t \text{ for all } t \in (0,1), x, y \in \mathbb{R}^n.$$

Edge expansion: Edge expansion of a graph $G = (V, E)$ is

$$\min\left\{\frac{e(A, V \setminus A)}{|A|} : A \subseteq V, 1 \leq |A| \leq \tfrac{1}{2}|V|\right\}$$

where $e(A, B)$ is the number of edges of G going between A and B.

1. Prove the Prékopa-Leindler inequality using the Prékopa-Leindler inequality with the additional assumption that $\sup h = \sup f = \sup g = 1$. **[3]**

   Nápověda: Consider $f'(x) = \frac{f(x)}{\sup(f)}$ and $g'(y) = \frac{g(y)}{\sup(g)}$. First show the case that $\sup f = \sup g = 1$, without the restriction on $\sup h$, implies the general Prékopa-Leindler inequality. Next show that $\sup h = \sup f = \sup g = 1$ implies the case $\sup f = \sup g = 1$. For this notice that h only has supremum at least 1, so we can truncate h if it goes higher, i.e. define $h'(x) = \min\{h(x), 1\}$.

2. Show that a positive function $f(x)$ is log-concave if and only if the function $\log(f(x))$ is concave. **[2]**

   Nápověda: Verify the definition.

3. Show that log-concave functions are closed under products, projections and convolutions.
   The projection of a function $f\colon \mathbb{R}^{m+n} \to \mathbb{R}$ is a function $g\colon \mathbb{R}^m \to \mathbb{R}$ defined as $g(x) := \int_{\mathbb{R}^n} f(x, y)\,\mathrm{d}y$, and the convolution is $(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y$ for $f, g\colon \mathbb{R}^n \to \mathbb{R}$. **[6]**

   Nápověda: The first part follows from the definition. For the second and third use Prékopa-Leindler inequality.

4. Let $\mathcal{G}$ be an (infinite) class of graphs such that its members have maximum degree at most d and edge expansion at least $c_0 > 0$. Show that there exists $c > 0$ such that for all $G \in \mathcal{G}$ the following holds

$$1 - \frac{|A_t|}{|V(G)|} \leq e^{-ct}$$

for every $A \subseteq V(G)$ such that $|A| \geq \frac{1}{2}|V(G)|$ and $t \geq 0$. A_t is the set of all vertices with distance at most t from some element of A (in particular $A_t \supseteq A$). **[4]**

Nápověda: Consider $B = V \setminus A_t$ and B_t.

5. Let $A, B \subseteq \mathbb{R}^d$ be convex sets. Prove that: **[3]**

$$\operatorname{conv}\left((\{0\} \times A) \cup (\{1\} \times B)\right) = \bigcup_{t \in [0,1]} \left(\{t\} \times ((1-t)A \oplus tB)\right)$$

Nápověda: If C and D are convex, then each point of $\operatorname{conv}(C \cup D)$ lies on some line joining a point from C and D.