Mathematics++

Problem set 2 – Measure and integral

hints after **April 4, 2022**, solutions due **April 11, 2022** before the lecture
Send your solutions to `chmel@kam.mff.cuni.cz`

1. Let $(X, \mathcal{S}, \mu)$ be a measurable space and $f, g : X \to \mathbb{R}$ be measurable functions. Prove that if $\int g < \infty$ and $|f| \leq g$ almost everywhere, then $\int f < \infty$. **[3]**

2. (a) Is there a function $f : [0,1] \to \mathbb{R}$ such that the Lebesgue integral $\int_0^1 f$ is finite (in particular it exists) but the Newton integral $\int_0^1 f$ does not exist? **[2]**

   (b) Is there a function $f : [0, \infty) \to \mathbb{R}$ such that the Newton integral $\int_0^\infty f$ is finite (in particular exists) but the Lebesgue integral $\int_0^\infty f$ does not exist? **[3]**

3. Construct a sequence of continuous function $f_n \colon [0,1] \to [0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 0$ but the sequence $(f_n(x))_n$ does not converge for any $x \in [0,1]$. **[4*]**

4. Show that the function $f \colon (0,1) \times (0,1) \to \mathbb{R}$ defined as

$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

   is not (Lebesgue) integrable on $(0,1) \times (0,1)$. **[4*]**

5. Let $f \colon \mathbb{R}^k \to \mathbb{R}$ be bounded Lebesgue mesuarable function. Prove that there are Borel functions $g, h \colon \mathbb{R}^k \to \mathbb{R}$ such that $g = h$ almost everywhere and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$. **[4]**

6. Prove or disprove the following claims:

   (a) Let $f : [0,1] \to \mathbb{R}$ be a non-negative, bounded and measurable function. Then,

$$\int_{[0,1]} f\,\mathrm{d}\mu = \inf \int_{[0,1]} \varphi\,\mathrm{d}\mu,$$

   where the infimum is taken over all simple measurable functions φ with $f \leq \varphi$. **[2]**

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, bounded and measurable function. Then,

$$\int_{\mathbb{R}} f\,\mathrm{d}\mu = \inf \int_{\mathbb{R}} \varphi\,\mathrm{d}\mu,$$

   where the infimum is taken over all simple measurable functions φ with $f \leq \varphi$. **[2]**
