Mathematics++

Problem set 1 – Measure and σ-algebras

hints after **March 14, 2022**, solutions due **March 21, 2022** before the lecture
Send your solutions to `chmel@kam.mff.cuni.cz`

Definition: Let $x \in \mathbb{R}$ and $E \subset \mathbb{R}$ be a measurable set. We define the **density** of E at x as the limit

$$d_E(x) := \lim_{\delta \to 0} \frac{\lambda((x-\delta, x+\delta) \cap E)}{2\delta}$$

if such limit exists.

1. Prove or disprove that there exists an infinite σ-algebra with countably many elements only. **[5*]**

   Hint: It does not exist. Show that every hypothetical countable σ-algebra is atomic. An atom is a minimal non-empty element, and a σ-algebra is atomic if every element can be written as a union of atoms.

2. Show that the union of two Lebesgue measurable sets is Lebesgue measurable. **[2]**

   Hint: Use the definition of measurable set twice.

3. Let $\mathcal{S}$ be a σ-algebra and let $\mu : \mathcal{S} \to \overline{\mathbb{R}_0^+}$ be additive (i.e., for any disjoint $A, B \in \mathcal{S}, \mu(A) + \mu(B) = \mu(A \cup B)$). Show that the following are equivalent:

   - μ is lower semi-continuous, that is, for any $B_1 \subset B_2 \subset \ldots \in \mathcal{S}$, we have $\mu(\cup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n)$,
   - μ is countably additive, that is, for any disjoint $A_1, A_2, \ldots \in \mathcal{S}$, we have $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ **[2]**

   Hint: $\Downarrow$: build B_i from A_i in the following way: $B_i := \bigcup_{j=1}^{i} A_j$.

   $\Uparrow$: build A_i from B_i as follows: $A_i := B_i \setminus B_{i-1}$ (and $A_1 = B_1$).

4. Let $(X, \mathcal{S}, \mu)$ be a measurable space and $\{A_i\}_{i=0}^{\infty}$ be a sequence of measurable sets such that $A_{i+1} \subseteq A_i$ for every i. Assuming $\mu(A_0) < \infty$, show that

   $$\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=0}^{\infty} A_i\right).$$

   In addition, show that the assumption above is necessary, that is, find a sequence as above which violates $\mu(A_0) < \infty$ as well as the conclusion. **[4]**

   Hint: You want to use σ-additivity, and hence you need to move from intersection to union.

5. Prove that the set C defined below is mesuarable and determine its (Lebesgue) measure.

   Let $\{\mathcal{K}_n\}$ be a sequence of finite collections of closed intervals defined inductively as

   - $\mathcal{K}_0 = \{[0, 1]\}, \mathcal{K}_1 = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$,

- $\mathcal{K}_n$ is obtained from $\mathcal{K}_{n-1}$ by removing the open middle third of each of the intervals in $\mathcal{K}_{n-1}$.

Then, we set $K_n := \bigcup \mathcal{K}_n$ and $C := \bigcap_n K_n$. **[4]**

Hint: Use the previous exercise.

6. Show that every measurable set of finite measure can be approximated with arbitrary precision by a finite union of intervals; that is, $\forall E \subset \mathbb{R}$ of finite measure and $\forall \varepsilon > 0$ there is $A \subset \mathbb{R}$, which is a union of finitely many open intervals, such that $\lambda(E \triangle A) \leq \varepsilon$.

   In addition, show that the assumption on finite measure is necessary; that is, find a measurable set of infinite measure which cannot be approximated by a finite union of intervals for some $\varepsilon > 0$. **[4]**

   Hint: Use the definition of the outer measure.

7. Prove that there is no measurable set $E \subseteq (0, 1)$ such that $\lambda(E) = 1/2$ and for every $x \in (0, 1)$ the density of E at x equals $1/2$. **[6*]**
   Hint: You may want to use the approximation from the problem above.

   Hint: By contradiction. Try to find the point with a different density by a recursive construction. In each step, you may want to find an interval, where E is dense (or sparse) and use recursion on such interval. The sought point can be found in the intersection of the intervals.