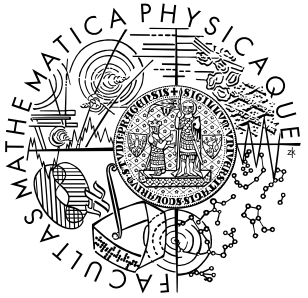
Proof-number search for combinatorial games



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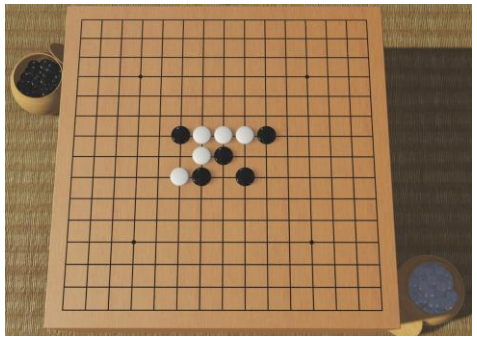
Solving combinatorial games

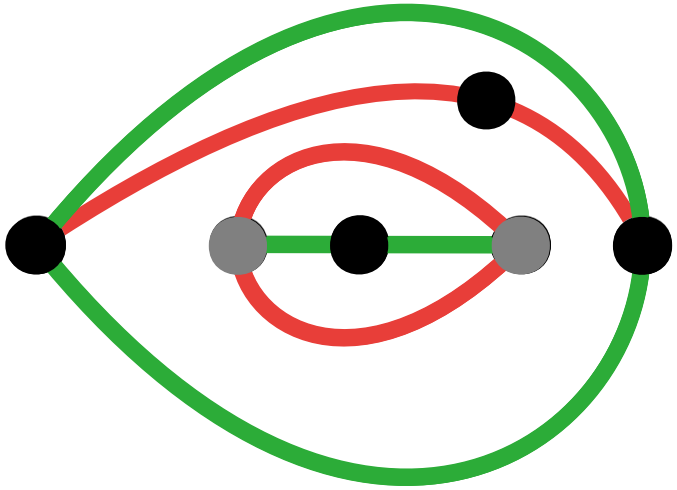
- Many combinatorial games have been weakly solved (Connect4, Gomoku, Checkers, Othello, etc.).
- Outcomes based on exhaustive search of game trees (Alpha-beta search, Proof-number search).

Sprouts game (Conway and Paterson, 1967)

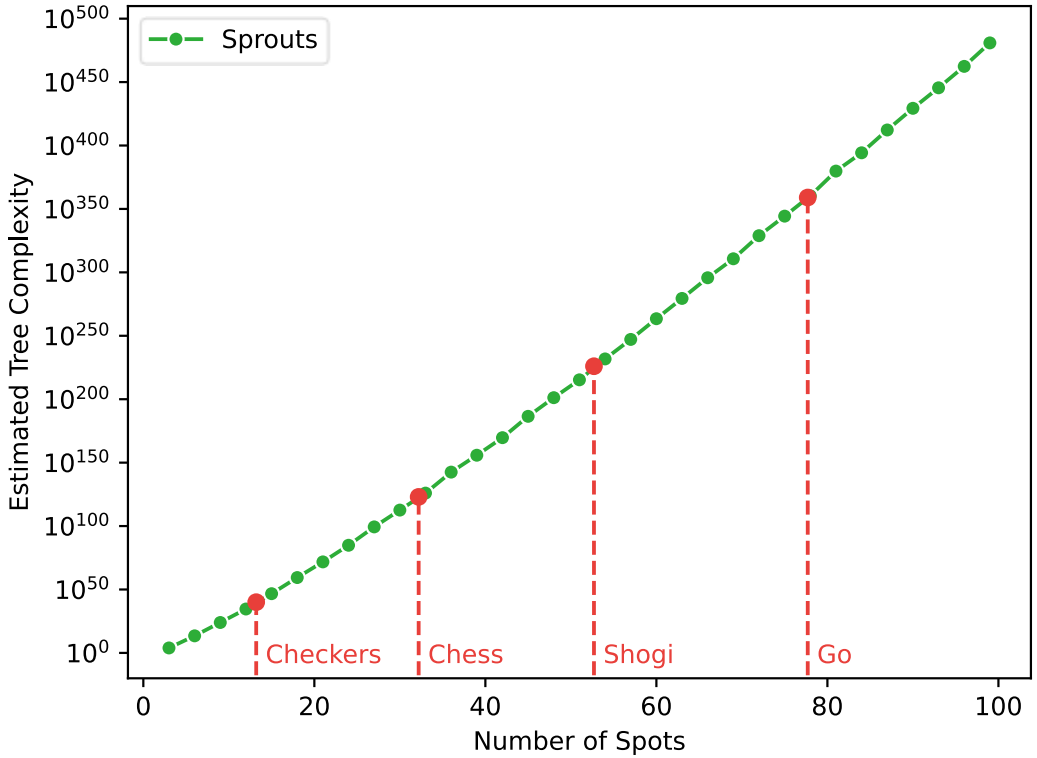
A combinatorial game with simple rules:

- Starts with *n* initial spots.
- Players alternate in connecting spots by curves (cycles are allowed).
- Curves cannot cross.
- A new spot is added along a newly drawn curve.
- Each spot can be incident to at most three curves.
- The first player with no move loses.



Sprouts properties

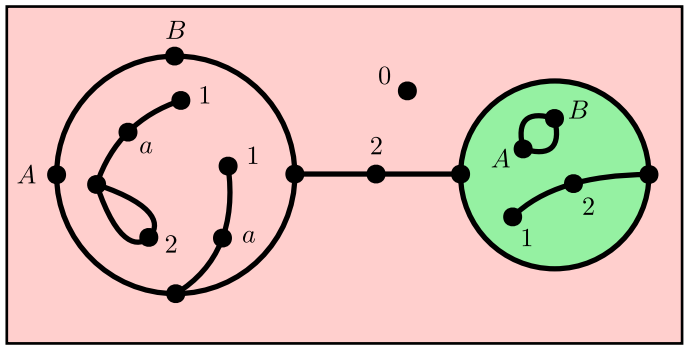
- The game trees of Sprouts grow extremely fast.
- Famous Sprouts conjecture: *n*-spot position is winning ↔ $n \equiv 3, 4, 5 \pmod 6$.

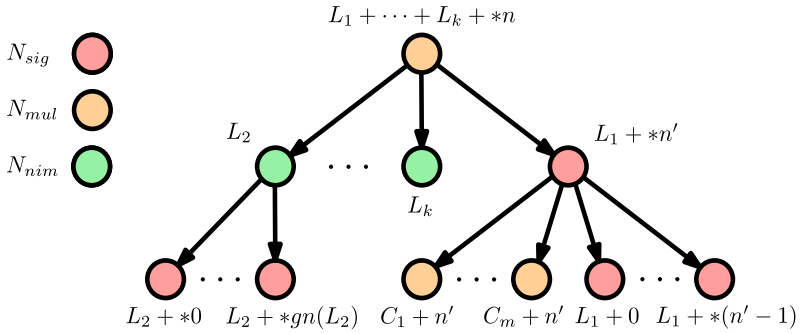


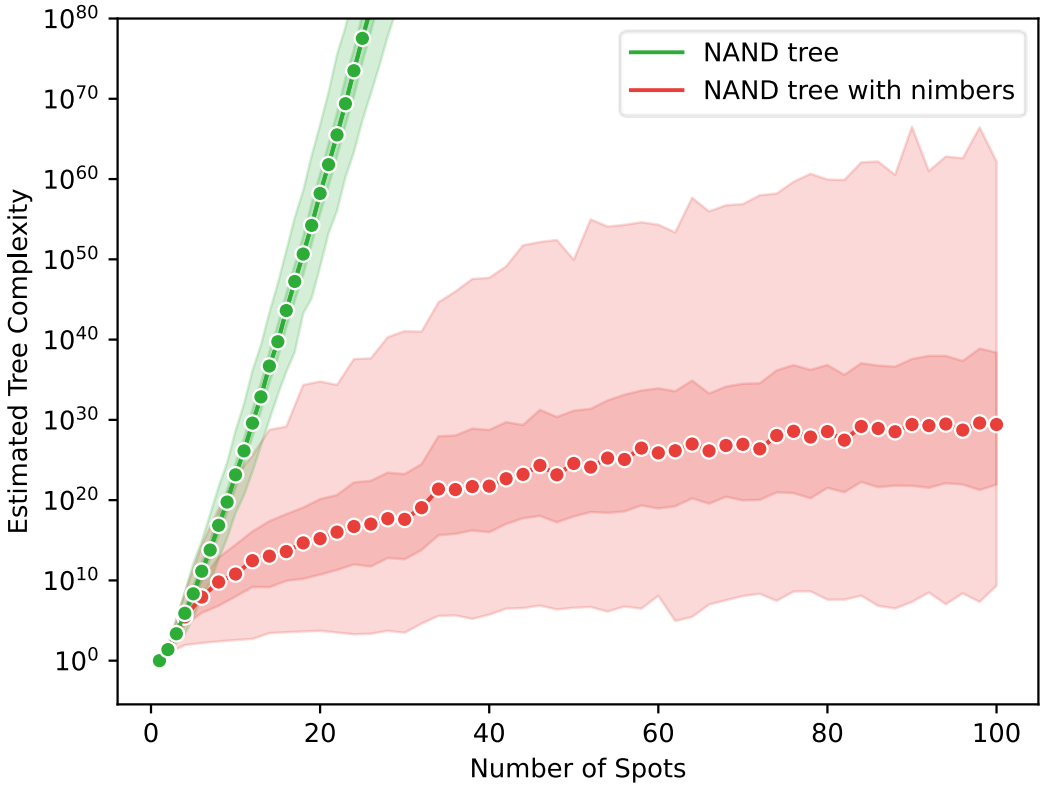
n	out.	auth.	n	out.	auth.	n	out.	auth.
1	L	Conway	19	L	Glop '07	37	L	Glop '10
2	L	Conway	20	L	Glop '07	38	L	Glop '10
3	W	Conway	21	W	Glop '07	39	W	Glop '11
4	W	Mollison	22	W	Glop '07	40	W	Glop '07
5	W	Mollison	23	W	Glop '07	41	W	Glop '07
6	L	Mollison	24	L	Glop '07	42	L	Glop '11
7	L	AJS	25	L	Glop '07	43	L	Glop '11
8	L	AJS	26	L	Glop '07	44	L	Glop '11
9	W	AJS	27	W	Glop '07	45	?	?
10	W	AJS	28	W	Glop '07	46	W	Glop '10
11	W	AJS	29	W	Glop '07	47	W	Glop '07
12	L	Purinton	30	L	Glop '07	48	?	?
13	L	Purinton	31	L	Glop '07	49	?	?
14	L	Purinton	32	L	Glop '07	50	?	?
15	W	Glop '07	33	W	Glop '10	51	?	?
16	W	Glop '07	34	W	Glop '07	52	?	?
17	W	Glop '07	35	W	Glop '07	53	W	Glop '10
18	L	Glop '07	36	L	Glop '10	54	?	?

Reducing game tree complexity

- Independent subgames of Sprouts can be analyzed separately using combinatorial game theory (Sprague-Grundy Theorem).

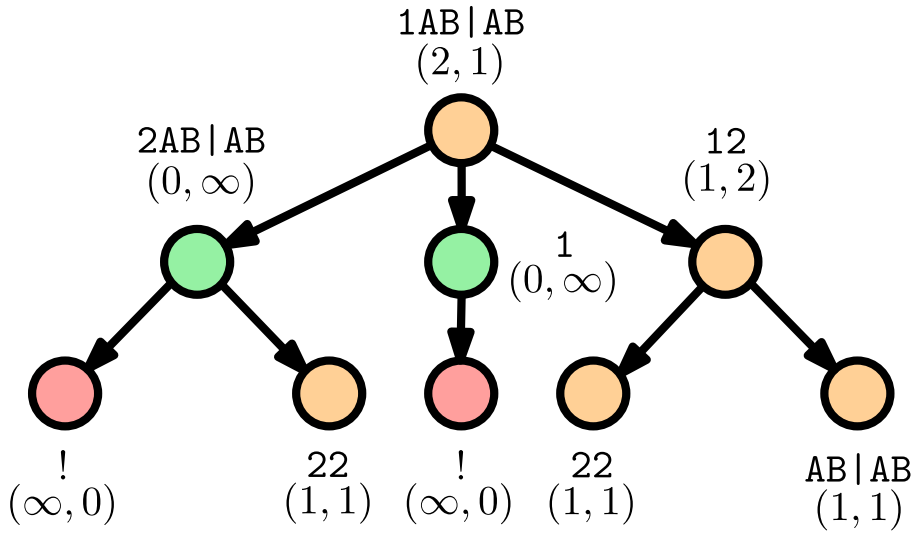






Proof-number search

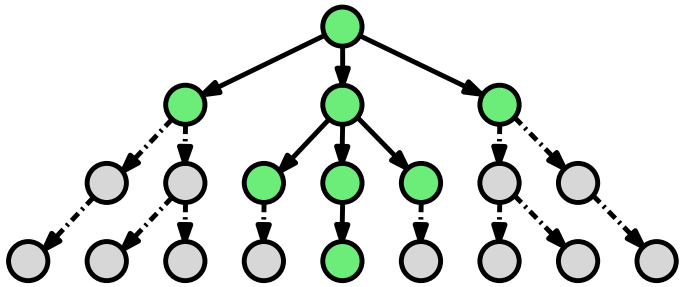
- Alpha-beta pruning can easily get stuck in difficult subtrees.
- Using Proof-number search, a best-first search algorithm focusing on potentially shortest proofs:
  - Each node *N* is associated with a proof number *pn(N)* and a disproof number *dn(N)*.
  - Initialized heuristically in leaves.
  - Aggregated from children in expanded nodes.
  - Expand the most-proving node (MPN) and update.



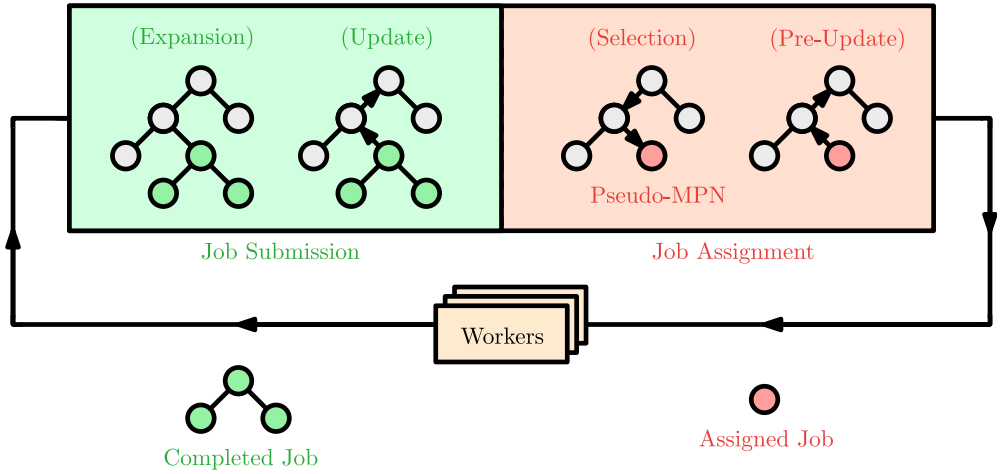
Our contribution

- We adapted dfpn (the standard variant of Proof-number search) for impartial games.
- We developed a new well-scaling parallel Proof-number search variant:
  - Targets large distributed-memory environments.
  - Shares results among workers.
  - Two-level parallelization.

→ A much faster solver SPOTS that allowed us to compute many new outcomes for Sprouts.



$$
\begin{aligned}
pt(N_{next}) &= dt(N_{curr}) - dn_0(N_{curr}) + pn_0(N_{next}),\\
dt(N_{next}) &= \min\{pt(N_{curr}), dn(N_{next_2}) + 1\},\\
mint(N_{next}) &= mint(N_{curr}),\\
pS(N_{next}) &= dS(N_{curr}) + dn_0(N_{curr}) - pn_0(N_{next}),\\
dS(N_{next}) &= pS(N_{curr}).\\
pt(N_{next}) &= dt(N_{next}) = \infty,\\
mint(N_{next}) &= \text{thMins}_{curr} - \text{otherMins}_{curr,next},\\
pS(N_{next}) &= dS(N_{next}) = 0,
\end{aligned}
$$



Resulting solver

- SPOTS roughly 2800x faster (480 cores) than Glop.

→ **1 day of SPOTS ≈ 8 years of Glop**

- Outcome of 47 *n*-spot positions known until now.

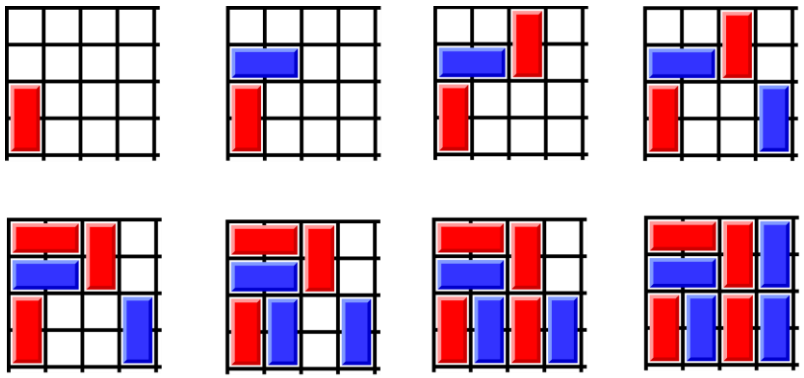
→ **We compute 42 new outcomes!**

  - The largest proof is 1000x larger than the largest so far (took 24 days to compute ≈ 280,000 CPU hours).
  - The Sprouts conjecture remains open.

n	out.	auth.	n	out.	auth.	n	out.	auth.
1	L	Conway	36	L	Glop '10	71	?	?
2	L	Conway	37	L	Glop '10	72	?	?
3	W	Conway	38	L	Glop '10	73	?	?
4	W	Mollison	39	W	Glop '11	74	?	?
5	W	Mollison	40	W	Glop '07	75	?	?
6	L	Mollison	41	W	Glop '07	76	?	?
7	L	AJS	42	L	Glop '11	77	?	?
8	L	AJS	43	L	Glop '11	78	?	?
9	W	AJS	44	L	Glop '11	79	?	?
10	W	AJS	45	?	?	80	?	?
11	W	AJS	46	W	Glop '10	81	?	?
12	L	Purinton	47	w	Glop '07	82	?	?
13	L	Purinton	48	?	?	83	?	?
14	L	Purinton	49	?	?	84	?	?
15	W	Glop '07	50	?	?	85	?	?
16	W	Glop '07	51	?	?	86	?	?
17	W	Glop '07	52	?	?	87	?	?
18	L	Glop '07	53	W	Glop '10	88	?	?
19	L	Glop '07	54	?	?	89	?	?
20	L	Glop '07	55	?	?	90	?	?
21	W	Glop '07	56	?	?	91	?	?
22	W	Glop '07	57	?	?	92	?	?
23	W	Glop '07	58	?	?	93	?	?
24	L	Glop '07	59	?	?	94	?	?
25	L	Glop '07	60	?	?	95	?	?
26	L	Glop '07	61	?	?	96	?	?
27	W	Glop '07	62	?	?	97	?	?
28	W	Glop '07	63	?	?	98	?	?
29	W	Glop '07	64	?	?	99	?	?
30	L	Glop '07	65	?	?	100	?	?
31	L	Glop '07	66	?	?	101	?	?
32	L	Glop '07	67	?	?	102	?	?
33	W	Glop '10	68	?	?	103	?	?
34	W	Glop '07	69	?	?	104	?	?
35	W	Glop '07	70	?	?	105	?	?

Future work

- Extend obtained results for other combinatorial games (Cram, Breakthrough).



Thank you for your attention.