Lecture notes for Discrete mathematics of Paul Erdős - NDMI107
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HAMILTON CYCLES

1 A theorem that involves degrees of vertices

During his visit to University of Waterloo in the fall of 1970, Erdős gave a lecture for its Department of Combinatorics and Optimization and presented there a proof of his fresh refinement of Turán's theorem:

Theorem 1.1 (Erdős [10])). *Let r be an integer greater than 1. For every graph G with $\omega(G) < r$, there is a graph H such that*

(i) G and H share their vertex-set V,

(ii) $d_G(v) \le d_H(v)$ for all v in V,

(iii) H is complete $(r-1)$-partite.

(The fourth property of H in Erdős's theorem, *if $d_G(v) = d_H(v)$ for all v in V, then $H = G$,* is an icing on the cake irrelevant to the discussion that follows.)

I was lucky to be in the audience. As I sat and listened, the beauty of this theorem took my breath away. Let me elaborate.

When G is a graph and r is a positive integer, the property that $\omega(G) \ge r$ can be certified by pointing out r pairwise adjacent vertices in G; finding such a certificate may be extremely difficult, but verifying it is straightforward and quick. By contrast, no easily verifiable certificate of the property that $\omega(G) < r$ is known.

To speculate about hypothetical certificates of $\omega(G) < r$, let us say that a graph G with $\omega(G) < r$ is *maximal* with respect to this property if adding any edge to G produces a graph H such that $\omega(H) \ge r$. The property that $\omega(G) < r$ could be certified by exhibiting a graph H that is maximal with respect to $\omega(G) < r$ and showing that G is a subgraph of H. However, this trick only shifts the burden of proof from all graphs with $\omega < r$ to all maximal graphs with $\omega < r$: how do we certify that $\omega(H) < r$? One way of doing this would be first to compile a catalogue of all maximal graphs with $\omega < r$ and then to simply point out H in this catalogue. Unfortunately, the catalogue is not only very large, but also very complex: some of its items H are wild in the sense that describing them is difficult. Yet the catalogue also includes items that are tame in the sense that describing them is easy: these tame items are the complete $(r-1)$-partite graphs.

Theorem 1.1 shows that the tame items play a special role in the catalogue: for every wild item G there is a tame item H such that $d_G(v) \le d_H(v)$ for all vertices v of these two graphs. This can be paraphrased as follows.

Theorem 1.2 (another form of Theorem 1.1)**.** *Let r be an integer greater than 1 and let $d_1, \ldots, d_n$ be an integer sequence. If there is no complete $(r-1)$-partite graph H with vertices $v_1, \ldots, v_n$ such that $d_H(v_i) \ge d_i$ for all i, then every graph G with vertices $v_1, \ldots, v_n$ such that $d_G(v_i) = d_i$ for all i has $\omega(G) \ge r$.*

If G is a graph with vertices $v_1, \ldots, v_n$, then the sequence $d_G(v_1), \ldots, d_G(v_n)$ is called the *degree sequence* of G. Theorem 1.2 describes a large set of integer sequences such that all graphs G with these degree sequences have $\omega(G) \ge r$, but it does not describe all of them. For example, if $3 \le r \le n-1$, then there is a unique graph G that has

$r-3$ vertices of degree $n-1$,
3 vertices of degree $r-1$,
$n-r$ vertices of degree $r-3$,

and this graph has $\omega(G) = r$, but Theorem 1.2 does not guarantee that $\omega(G) \ge r$: the complete $(r-1)$-partite graph $K(n-r+1, 2, 1, \ldots, 1)$ has

$r-3$ vertices of degree $n-1$,
2 vertices of degree $n-2$,
$n-r+1$ vertices of degree $r-1$.

Nevertheless, Theorem 1.2 is the best theorem of its kind.

To explain in what sense it is the best, let us say that a sequence $e_1, \ldots, e_n$ *majorizes* a sequence $d_1, \ldots, d_n$ if, and only if, $e_i \ge d_i$ for all i. Theorem 1.2 asserts that $\omega(G) \ge r$ if the degree sequence of G is not majorized by by a degree sequence of any complete $(r-1)$-partite graph. Additionally, let us say that a property of degree sequences is *monotone* if, with every degree sequence d that has this property, all degree sequences that majorize d have the property, too. Trivially, a monotone property of degree sequences forces $\omega(G) \ge r$ only if the degree sequence of G is not majorized by by a degree sequence of any complete $(r-1)$-partite graph. It follows that all theorems asserting that some monotone property of degree sequences forces $\omega(G) \ge r$ are subsumed in Theorem 1.2.

To put this observation in different terms, let $\mathcal{D}_n$ denote the set of all degree sequences of graphs on n vertices and let $\Omega_{n,r}$ denote the set of all d in $\mathcal{D}_n$ for which every G with degree sequence d has $\omega(G) \ge r$. A subset $\mathcal{U}$ of $\mathcal{D}_n$ is said to be *upward closed* if

$$d \in \mathcal{U},\ e \in \mathcal{D}_n,\ e \text{ majorizes } d \ \Rightarrow\ e \in \mathcal{U}.$$

Since the union of two upward closed sets is upward closed, the union of all upward closed subsets of any set $\mathcal{S}$ of degree sequences is upward closed; let $\mathcal{S}^{\uparrow}$ denote this largest upward closed subset of $\mathcal{S}$. (The degree sequence in the preceding example, $r-3$ terms $n-1$, 3 terms $r-1$, $n-r$ terms $r-3$, belongs to $\Omega_{n,r} - \Omega_{n,r}^{\uparrow}$.) Theorem 1.2 decribes $\Omega_{n,r}^{\uparrow}$ with $n \ge r \ge 2$: if a degree sequence fails to satisfy its condition, then it is majorized by a degree sequence outside

$\Omega_{n,r}$, and so it does not belong to $\Omega^{\uparrow}_{n,r}$. What excited me most about Erdős's Theorem 1.1 was that it blazed a trail: it displayed a template for theorems that infer properties of graphs from properties of their degree sequences. For instance, a graph is called *hamiltonian* if it contains a *Hamilton cycle,* which means a cycle passing through all its vertices.

The property of being hamiltonian is like the property $\omega(G) \geq r$ in several respects. The claim that a graph G is hamiltonian can be certified by pointing out a Hamilton cycle in G; finding such a certificate may be extremely difficult, but verifying it is straightforward and quick. By contrast, no easily verifiable certificate of the claim that G is nonhamiltonian is known. Let us say that a nonhamiltonian graph is *maximal* with respect to this property if adding any edge to this graph produces a hamiltonian graph. The claim that G is nonhamiltonian could be certified by exhibiting a maximal nonhamiltonian graph H and showing that G is a subgraph of H. However, this trick only shifts the burden of proof from all nonhamiltonian graphs to all maximal nonhamiltonian graphs: how do we certify that H is nonhamiltonian? One way of doing this would be first to compile a catalogue of all maximal nonhamiltonian graphs and then to simply point out H in this catalogue. Unfortunately, the catalogue is not only very large, but also very complex: some of its items H are wild in the sense that describing them is difficult.

My doctoral advisor Crispin Nash-Williams (1932–2001) coined in [13] the term *forcibly hamiltonian* for degree sequences d such that every G with degree sequence d is hamiltonian; by 1970, there was a progression of theorems ([9, Theorem 3], [15], [2]) that described larger and larger upward closed sets of forcibly hamiltonian sequences; having heard Erdős's lecture, I began to wonder if there was a neat description of the largest upward closed set of forcibly hamiltonian sequences.

Let us say that a degree sequence e *strictly majorizes* a degree sequence d if e majorizes d and $e \neq d$; let us say that a graph G is *degree-maximal* with respect to some property if G has the property and no graph with a degree sequence that strictly majorizes the degree sequence of G has the property. Erdős's Theorem 1.1 amounts to a neat catalogue of all degree-maximal graphs G with $\omega(G) < r$; I was led to look for a catalogue of all degree-maximal nonhamiltonian graphs G.

During the Christmas break, I compiled the catalogue of all degree-maximal nonhamiltonian graphs G with n vertices for $n = 3, 4, 5, 6, 7$ and then I saw a pattern: just like in Erdős's prototype, graphs in the catalogue were tame in the sense of being easily described. In each of them, the vertex set could be split into three pairwise disjoint nonempty parts A, B, C such that $|A| = |B|$; every vertex in A was adjacent to all the remaining vertices, the vertices in C were pairwise adjacent, and there were no other edges. (To see that every such graph is nonhamiltonian, observe that the removal of A breaks it into $|A| + 1$ connected components, namely, the $|A|$ isolated vertices in B and the nonempty clique C. This could never happen if the graph contained a Hamilton cycle: the removal of A would break the cycle into at most $|A|$ segments and these segments would hold the rest of the graph in at most $|A|$ pieces.)

In the notation of [4], each of these graphs can be specified as $K_k \vee (\overline{K_k} + K_{n-2k})$ for some positive integer k less than $n/2$. Here, $G+H$ denotes the *disjoint union* of G and H, which is the graph consisting of a copy of G and a copy of H that have no vertices in common; $G \vee H$ denotes the *join* of G and H, which is $G+H$ with additional edges that join every vertex in the copy of G to every vertex in the copy of H. In this $K_k \vee (\overline{K_k} + K_{n-2k})$, the vertex set of the K_k is A, the vertex set of the $\overline{K_k}$ is B, and the vertex set of the K_{n-2k} is C.

Now I knew what I had to prove:

Theorem 1.3. *Let n be an integer at least 3. For every nonhamiltonian graph G on n vertices, there is a graph H such that*

(i) G and H share their vertex-set V,

(ii) $d_G(v) \leq d_H(v)$ for all v in V,

(iii) $H = K_k \vee (\overline{K_k} + K_{n-2k})$ for some positive integer k less than $n/2$,

(A fourth property of H, *if $d_G(v) = d_H(v)$ for all v in V, then $H = G$,* is trivial since this H does not share its degree sequence with any other graph.)

Theorem 1.3 can be paraphrased as follows.

> *Let n be an integer at least 3. If the degree sequence of a graph G with n vertices is not majorized by the degree sequence of any $K_k \vee (\overline{K_k} + K_{n-2k})$ with $1 \leq k < n/2$, then G is hamiltonian.*

Since $K_k \vee (\overline{K_k} + K_{n-2k})$ has

k vertices of degree k,
$n-2k$ vertices of degree $n-k-1$,
k vertices of degree $n-1$,

the condition that the degree sequence of G is not majorized by the degree sequence of any such graph with $1 \leq k < n/2$ can be stated more directly:

Theorem 1.4 (another form of Theorem 1.3)**.** *Let n be an integer at least 3. If G is a graph with n vertices that, for each positive integer k less than $n/2$, has fewer than k vertices of degree at most k or fewer than $n-k$ vertices of degree at most $n-k-1$, then G is hamiltonian.*

I proved Theorem 1.4 in January 1971 and submitted the resulting paper [5] for publication on February 1.

Incidentally, this theorem describes a large set of forcibly hamiltonian sequences, but it does not describe all of them. For example, Nash-Williams [13] proved that, for every choice of positive integers k and n such that k is less than $n/2$ and even, every sequence of k terms equal to k and $n-k$ terms equal to $n-k-1$ is forcibly hamiltonian.

My proof of Theorem 1.4 was nonconstructive: it consisted of showing that for every maximal nonhamiltonian graph G on n vertices there is a positive integer k less than $n/2$ such that G has at least k vertices of degree at most k and at least $n-k$ vertices of degree at most $n-k-1$.

Three years and a few months after this, Adrian Bondy visited me in Montreal and complained about a student of his, who made no progress toward solving an easy problem that Adrian had proposed. The problem was to convert my proof into an efficient algorithm that, given a graph G satisfying the hypothesis of Theorem 1.4, returns a Hamilton cycle in G. I commiserated with Adrian but, as we talked about it, it began to dawn on us that the student may have not been all that weak and that the problem may have not been all that easy: we ourselves could not do it. Fortunately, this sorry state of affairs did not last long. Eventually we designed the algorithm that Adrian wanted and then we wrote it up, along with a plethora of generalizations, in [3].

1.1 An algorithmic proof of Theorem 1.4

Our starting point (which I had also used in my proof of Theorem 1.4) was a proof of the following theorem by Øystein Ore (1899–1968):

Theorem 1.5 (Ore [14])**.** *Let G be a graph of order n and let u, v be distinct nonadjacent vertices of G such that G with edge uv added is hamiltonian. If $d_G(u) + d_G(v) \geq n$, then G is hamiltonian.*

Proof. By assumption, G with edge uv added contains a cycle $u_1u_2 \ldots u_nu_1$. If none of its n edges is uv, then this cycle is a Hamilton cycle of G and we are done; else we may assume, without loss of generality, that $uv = u_nu_1$. Write

$$S = \{i : u_1 \text{ is adjacent to } u_{i+1}\},$$
$$T = \{i : u_n \text{ is adjacent to } u_i\}$$

and note that S and T are subsets of $\{1, 2, \ldots, n-1\}$. If $d_G(u) + d_G(v) \geq n$, then, since $|S| = d_G(u_1)$ and $|T| = d_G(u_n)$, we have $S \cap T \neq \emptyset$. With i standing for any subscript in $S \cap T \neq \emptyset$,

$$u_1u_{i+1}u_{i+2} \ldots u_n, u_iu_{i-1}u_1$$

is a Hamilton cycle in G. □

Now imagine looking for a Hamilton cycle in a prescribed graph G. If the hypothesis of Theorem 1.5 is satisfied, then we may look instead for a Hamilton cycle in the graph H arising from G by adding edge uv: the proof of the theorem shows how a Hamilton cycle using this edge can be transformed into a Hamilton cycle not using it. Furthermore, if the new graph H and some other pair of distinct nonadjacent vertices x, y satisfy $d_H(x) + d_H(y) \geq n$, then we may in turn augment H by adding edge xy. Repeating this process as long as we can, we eventually arrive at a graph which we call the *closure of* G. (Here, the definite article is apt: it is easy to prove that the closure is unique, independent of the order in which new edges are added to the input graph. However, the argument is irrelevant to the discussion that follows and we will not let it distract us.) To be able to reverse the construction later on, we record with each new edge the time when it gets added to the emerging closure.

$H = G$, `timestamp`$(e) = 0$ for all edges e of G, `maxtimestamp` $= 0$;
while H has distinct nonadjacent vertices u, v
such that $d_H(u) + d_H(v) \geq n$
do add edge uv to H, `timestamp`$(uv) =$ `maxtimestamp` $+1$,
`maxtimestamp` $=$ `timestamp`(uv);
end

Next, we propose to prove that, as long as the hypothesis of Theorem 1.4 is satisfied, the closure H of G is a complete graph: assuming that H is not complete, we will produce a positive integer k less than $n/2$ such that

(i) $0 < k < n/2$,
(ii) at least k vertices w of H have $d_H(w) \leq k$,
(iii) at least $n - k$ vertices w of H have $d_H(w) \leq n - k - 1$.

Since $d_G(w) \leq d_H(w)$ for all w, properties (i), (ii), (iii) imply that the hypothesis of Theorem 1.4 is not satisfied.

In producing k, we may assume that H has no isolated vertices: else (i), (ii), (iii) are satisfied by setting $k = 1$. Under this assumption, let v denote a vertex maximizing $d_H(w)$ among all w such that $d_H(w) \leq n - 2$ (there are such vertices since $H \neq K_n$) and let u denote a vertex maximizing $d_H(w)$ among all w nonadjacent to v in H. We are going to show that (i), (ii), (iii) are satisfied by setting $k = d_H(u)$.

Since u is not an isolated vertex, we have $d_H(u) > 0$. The stopping condition of the **while** loop in the construction of H guarantees that $d_H(u) + d_H(v) \leq n - 1$ and our choice of v guarantees that $d_H(v) \geq d_H(u)$; it follows that $d_H(u) < n/2$. Now (i) is verified. Vertex v is nonadjacent to $n - 1 - d_H(v)$ vertices other than itself; our choice of u guarantees that each of these vertices has degree at most $d_H(u)$; since $d_H(v) \leq n - 1 - d_H(u)$, their number is at least $d_H(u)$. Now (ii) is verified. Vertex u is nonadjacent to $n - 1 - d_H(u)$ vertices other than itself; our choice of v guarantees that each of these vertices has degree at most $d_H(v)$, which is at most $n - 1 - d_H(u)$; in addition to these $n - 1 - d_H(u)$ vertices, u also has degree at most $n - 1 - d_H(u)$. Now (iii) is verified.

Finally, any Hamilton cycle C in the closure of G can be transformed into a Hamilton cycle C in G by iterating the argument that proves Theorem 1.5:

uv = edge of C that maximizes $\texttt{timestamp}(uv)$;
while $\texttt{timestamp}(uv) > 0$
do list the vertices of C in their cyclic order as
  $u_1, u_2, \ldots, u_n, u_1$ with $uv = u_1u_n$;
 find a subscript i such that
  $\texttt{timestamp}(u_1u_{i+1}) < \texttt{timestamp}(uv)$ and
  $\texttt{timestamp}(u_nu_i) < \texttt{timestamp}(uv)$;
 C = the Hamilton cycle $u_1u_{i+1}u_{i+2}\ldots u_n, u_iu_{i-1}u_1$;
 uv = edge of C that maximizes $\texttt{timestamp}(uv)$;
end

With each iteration of the **while** loop, the largest $\texttt{timestamp}(e)$ with e running through all the edges of C drops, and so the loop eventually terminates.

1.2 A digression: Testing the hypothesis of Theorem 1.2

It is obvious how to test the hypothesis of Theorem 1.4, but it it is not obvious how to test the hypothesis of Theorem 1.2. With $k_{\min}(d_1, \ldots, d_n)$ standing for the smallest k such that some complete k-partite graph H with vertices $v_1, \ldots, v_n$ has $d_H(v_i) \geq d_i$ for all i, the hypothesis of Theorem 1.2 can be stated as

$$r - 1 < k_{\min}(d_1, \ldots, d_n).$$

Owen Murphy [12] designed an efficient algorithm that, given an integer sequence d_1, d_2,…,d_n such that $0 \leq d_1 \leq \ldots \leq d_n$, returns the smallest k such that some graph G with vertices $v_1, \ldots, v_n$ (i) consists of k pairwise vertex-disjoint cliques and (ii) has $d_G(v_i) \leq d_i$ for all i. Since G has property (i) if and only if its complement is complete k-partite, we have

$$k = k_{\min}(n - 1 - d_n,\ \ldots, n - 1 - d_1),$$

and so converting Murphy's algorithm into an algorithm for computing $r_{\min}(d_1, \ldots, d_n)$ is a matter of mechanical routine.

Let us begin our discussion of the converted version with the following lemma:

Lemma 1.1. *For every integer sequence $d_1, \ldots, d_n$ such that $d_1 \leq \ldots \leq d_n \leq n - 1$ there is a complete k-partite graph H with vertices $v_1, \ldots, v_n$ such that*
- *(i)* $k = k_{\min}(d_1, \ldots, d_n)$,
- *(ii)* $d_H(v_i) \geq d_i$ *for all* $i = 1, \ldots, n$,
- *(iii) one of the k parts of H is $\{v_{a+1}, v_{a+2}, \ldots, v_n\}$, where $a = d_n$.*

Proof. Consider an arbitrary graph H with properties (i), (ii) and let S denote the part of H that includes v_n. Since $d_H(v_n) = n - |S|$, we have $|S| \leq n - d_n$. If

$|S| < n - d_n$, then transfer any $n - d_n - |S|$ vertices from the outside of S into S. This transformation maintains property (ii): after the move, all the vertices in S have degree d_n and the degrees of all the other vertices remain unchanged or increase during the move. Since the transformation maintains property (ii) and does not increase the number of parts of H, it maintains property (i) as well. Now $|S| = n - d_n$. Finally, if there are subscripts i and j such that $1 \leq i < j \leq n$ and $v_i \in S$, $v_j \notin S$, then $d_H(v_i) = d_n \geq d_j$, $d_H(v_j) \geq d_j \geq d_i$, and so swapping the labels of these two vertices maintains property (ii); trivially, it maintains property (i) as well. Repeating this operation until no such pair v_i, v_j is present any more produces an H with all three properties (i), (ii), (iii). □

Lemma 1.1 points out the following recursive algorithm that, given any integer sequence $d_1, d_2, \ldots, d_n$ such that $d_1 \leq \ldots d_n \leq n-1$, returns $k_{min}(d_1, \ldots, d_n)$.

if $d_n \leq 0$
then **return** 1;
else $s = n - d_n$;
  return $1 + k_{min}(d_1 - s, \ldots, d_{n-s} - s)$;
end

Here is the same algorithm presented in an iterative form (chosen to resemble the iterative algorithm on page 209 of [12]):

$n_0 = n$, $k = 0$;
while $n_k > 0$
do $t = n - n_k$;
  $n_{k+1} = d_{n-t} - t$;
  $k = k + 1$;
end
return k;

For instance, given the sequence $1, 1, 1, 2, 3, 3, 5$, the algorithm proceeds as follows:

recursive form		iterative form
$k_{min}(1,1,1,2,3,3,5)$	$[s=2]$	$n_0 = 7,\ k = 0$
$= 1 + k_{min}(-1,-1,-1,0,1)$	$[s=4]$	$t = 0,\ n_1 = 5,\ k = 1$
$= 2 + k_{min}(-5)$		$t = 2,\ n_2 = 1,\ k = 2$
$= 3$		$t = 6,\ n_3 = -5,\ k = 3$

2 A theorem that involves connectivity and stability

As we have observed, the graphs $K_k \vee (\overline{K_k} + K_{n-2k})$ with $1 \leq k < n/2$ that are featured in Theorem 1.3 are nonhamiltonian for a simple reason: each of them contains a nonempty set A of vertices, whose removal breaks the rest of the graph into more than $|A|$ connected components. Let us say that a graph is *tough* if it does not contain any such set A. In this terminology, every hamiltonian graph is tough, but the converse is false: for instance, the graph with vertices $u_1, u_2, u_3, v_1, v_2, v_3, w$ and edges

$$u_1u_2,\ u_1u_3,\ u_2u_3,\ u_1v_1,\ u_2v_2,\ u_3v_3,\ u_1w,\ u_2w,\ u_3w,\ v_1w,\ v_2w,\ v_3w$$

is tough but nonhamiltonian. (In [6], I conjectured that a weaker version of the converse might be true, though: I proposed to call a graph t-tough, with t a positive number, if it contains no nonempty set A of vertices, whose removal breaks the rest of the graph into more than $t|A|$ connected components and I conjectured the existence of a positive number t_0 such that every t_0-tough graph is hamiltonian. This conjecture remains open; Doug Bauer, Hajo Broersma, and Hendrik Veldman [1] proved that every such t_0 is at least 9/4. For more on the conjecture, see [11].)

Having proved Theorem 1.3, I was drawn to musings on which graphs are tough and which graphs are not. Maximal non-tough graphs are easily described: each of them is the join of a complete graph with some positive number k of vertices and a graph that is the disjoint union of $k+1$ nonempty complete graphs. (These graphs include not only all the graphs featured in Theorem 1.3, but other graphs as well. For instance, $K_1 \vee (K_2 + K_2)$ is a maximal non-tough graph, which does not appear in Theorem 1.3, since its degree sequence is stricly majorized by the degree sequence of another maximal non-tough graph, $K_2 \vee (K_1 + K_1 + K_1)$.) The shape of these graphs suggests a relationship between toughness and two classic invariants of graphs: the *stability number* $\alpha(G)$, defined as the largest number of pairwise disjoint vertices in G, and the *connectivity* $\kappa(G)$, defined as the smallest number of vertices whose removal breaks the rest of G into at least two connected components. (In particular, $\kappa(G) = 0$ means that G is disconnected. Note also that this definition does not specify connectivity of complete graphs; that is declared as $\kappa(K_n) = n - 1$.) More precisely, if G is not tough, then $\alpha(G) > \kappa(G)$: when the removal of a nonempty set A of vertices breaks the rest of the graph into more than $|A|$ connected components, we have $\kappa(G) \leq |A|$ and (as vertices in different components of G with A removed are nonadjacent) $\alpha(G) > |A|$. So now I had two different conditions implying that a graph G is tough: one was that G is hamiltonian and the other was that $\alpha(G) \leq \kappa(G)$. But I could not find any graph that would satisfy the latter condition without satisfying the former.

In the spring of 1971, I traveled through the West Coast of Canada and United States: I had hiring interviews in Victoria and at Stanford, I gave talks at the University of Washington in Seattle and at University of California at Los Angeles, I went to San Francisco just to see it and I visited the RAND Corporation in Santa Monica. All this time, I kept switching back and forth between two distinct modes of operation: when a university paid for a particular leg of the trip, I flew on airplanes and slept in three-star hotels; when I was on my own, I hitchhiked and stayed at the YMCA. I incorporated into this tour a number theory conference held in March at the Washington State University since Paul Erdős was among its participants.

When the conference ended, Richard Guy (1916–2020) and his wife Louise (1918–2010) were about to drive the PGOM to Calgary and offered to take me with them as far as I wanted to go. Just before we got into the car, I mentioned to Erdős my suspicion that the condition $\alpha(G) \le \kappa(G)$ implied that G was hamiltonian. This stratagem earned me a place beside him in the back seat and his undivided attention for the duration of my ride with them. In just under two hours, we arrived in Spokane and by that time Erdős had explained to me a proof that my suspicion was justified. Then they dropped me off and my luck continued: within minutes I hitched a ride with a genial truck driver, who took me on the I-90 all the way to Seattle.

When I wrote it up, I added a footnote that read

> *This note was written in Professor Richard K. Guy's car on the way from Pullman to Spokane, Wash. The authors wish to express their gratitude to Mrs. Guy for smooth driving.*

Erdős liked the footnote and I was glad.

Theorem 2.1 ([7])**.** *If G is a graph with at least 3 vertices such that $\alpha(G) \le \kappa(G)$, then G is hamiltonian.*

Proof. A *vertex cut* is a set of vertices whose removal leaves the rest of the graph disconnected. We will specify an efficient algorithm that, given any graph G with at least 3 vertices, returns either a vertex cut K and a stable set A such that $|A| > |K|$ (which certifies that the hypothesis of the theorem is false) or else a Hamilton cycle in G (which certifies that the conclusion of the theorem is true).

In its simple preliminary phase, the algorithm produces either a vertex cut $\{w\}$ and a stable set $\{u, v\}$ (in which case it terminates) or else a cycle in G (in which case it proceeds to the subsequent main phase). The former outcome occurs when some vertex u has degree at most 1: vertex v is an arbitrary vertex nonadjacent to u and vertex w is either the unique neighbour of u or, if u is an isolated vertex, any vertex other than u. The latter outcome occurs when every vertex of G has degree at least 2. In this case, we build iteratively longer and longer paths $u_1u_2u_3\ldots$ starting from an arbitrary vertex u_2 and its distinct neighbours u_1, u_3. Once a path $u_1u_2\ldots u_k$ has been built, consider a neighbour v of u_k other than u_{k-1}. If $v = u_i$ for some i such that $1 \le i \le k-2$, then the preliminary phase terminates with cycle $u_iu_{i+1}\ldots u_ku_i$; else we set $u_{k+1} = v$ and proceed to the next iteration.

The main phase of the algorithm is also iterative. Each of its iterations begins with a cycle in G (in particular, the first iteration begins with the cycle produced in the preliminary phase). Let C denote this cycle.

Case 1: *C is not a Hamilton cycle.*
In this case, choose one of the two cyclic orientations of C and for each vertex v of C let v++ denote the immediate successor of v in this orientation. Let Q denote an arbitrary connected component of the graph arising from G by removing all vertices of C (and all edges incident with these vertices). Set

$$X = \{v : v \text{ has a neighbour in } Q\},$$
$$Y = \{v : v\text{++} \in X\}.$$

Subcase 1.1: *$X \cap Y = \emptyset$ and Y is a stable set.*
In this subcase, return the vertex cut X and the stable set $Y \cup \{u\}$ with u an arbitrary vertex in Q.

Subcase 1.2: *There is a vertex v such that $v \in X$ and v++$\in X$.*
In this subcase, v and v++ are joined by a path of length at least two that has all interior vertices in Q. Replace edge vv++ of C by this path and proceed to the next iteration with the longer cycle.

Subcase 1.3: *There is an edge v++w++ such that $v, w \in X$ and $w \neq v$++.*
In this subcase, replace edges vv++ and ww++ of C by edge v++w++ and a path between v to w with all interior vertices in Q; proceed to the next iteration with the longer cycle.

Case 2: *C is a Hamilton cycle.*
Return C. □

Which graphs satisfy the hypothesis of Theorem 2.1?

Proposition 2.1. *With $m^\star(n)$ standing for the smallest number of edges in a graph G with n vertices such that $\alpha(G) \le \kappa(G)$, we have*

$$\frac{1}{2}n^{3/2} - \frac{1}{4}n \ < \ m^\star(n) \ \le \ \frac{1}{2}n^{3/2} + \frac{5}{2}n\,.$$

Proof. First, we will prove that every graph G with n vertices and m edges which is not complete has

$$\alpha(G) \ge n^2/(2m+n), \tag{1}$$
$$\kappa(G) \le 2m/n. \tag{2}$$

To prove (1), we appeal to the bound

$$\mathrm{ex}(K_r, n) \;\le\; \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} \tag{3}$$

which follows from formula

$$\mathrm{ex}(K_r, n) \;=\; \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} - \frac{b(r-1-b)}{2(r-1)} \quad \text{where } b = n \bmod (r-1).$$

If r is an integer such that $r \ge 2$ and if number of edges of the complement $\overline{G}$ of G (which equals $\binom{n}{2} - m$) exceeds the right-hand side of (3), then $\overline{G}$ contains the complete graph K_r with r vertices. This fact may be recorded as

$$\binom{n}{2} - m > \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} \;\Rightarrow\; \alpha(G) \ge r$$

or, after simplifications, as

$$r - 1 < n^2/(2m+n) \;\Rightarrow\; \alpha(G) \ge r,$$

which is logically equivalent to (1). To prove (2), let v denote a vertex of G that has the smallest degree. Since G is not complete, some vertex w is nonadjacent to v and distinct from it, and so the set K of all the neighbours of v is a vertex cut separating v from w. Since the average degree of a vertex in G is $2m/n$, we have $|K| \le 2m/n$.

Under the assumption that $\alpha(G) \le \kappa(G)$, bounds (1) and (2) imply that

$$4m^2 + 2mn \;\ge\; n^3;$$

since the left-hand side of this inequality is an increasing function of m and its value at $\frac{1}{2}n^{3/2} - \frac{1}{4}n$ equals $n^3 - \frac{1}{4}n^2$, the lower bound on $m^\star(n)$ follows.

To establish the upper bound on $m^\star(n)$, we will construct, for an arbitrary positive integer n, a graph G with n vertices and at most $\frac{1}{2}n^{3/2} + \frac{5}{2}n$ edges such that $\alpha(G) \le \kappa(G)$. For this purpose, set $t = \lfloor n^{1/2} \rfloor$. For the vertex set of G, we take the union of t pairwise disjoint sets $V_1, V_2, \ldots, V_t$ whose sizes differ from each other by at most 1: each $|V_i|$ is $\lfloor n/t \rfloor$ or $\lceil n/t \rceil$. Since $t \le n^{1/2}$, we have $n/t \ge t$, and so in each V_i we can choose pairwise distinct vertices $v_i^1, v_i^2, \ldots v_i^t$. For each choice of i and j such that $1 \le i \le t-1$ and $1 \le j \le t$, we join vertex v_i^j to vertex v_{i+1}^j by an edge; for each i such that $1 \le i \le t$, we join every two vertices in V_i by an edge; apart from these, G has no other edges.

Since $n < (t+1)^2$, we have $n/t \le t+2$, and so each $|V_i|$ is at most $t+2$, and so the number m of edges of G satisfies $m \le t(t-1) + t\binom{t+2}{2} = \frac{1}{2}t^3 + \frac{5}{2}t^2 \le \frac{1}{2}n^{3/2} + \frac{5}{2}n$. Since the vertex set of G is covered by the t cliques V_i, we have $\alpha(G) \le t$. To see that $\kappa(G) \ge t$, observe that no set K of $t-1$ vertices of G can be a vertex cut: since the t paths $v_1^j v_2^j \ldots v_t^j$ with $j = 1, 2, \ldots, t$ are pairwise vertex-disjoint,

K cannot meet all of them. It follows that the graph $G - K$ arising from G by the deletion of all vertices in K (and all edges incident with these vertices) is connected: it contains at least one of the paths $v_1^j v_2^j \ldots v_t^j$ and, for each $i = 1, 2, \ldots, t$, all of its vertices in V_i are adjacent to v_i^j. □

Proposition 2.1 shows that the hypothesis of Theorem 2.1 can be satisfied by relatively sparse graphs. Graphs that satisfy the hypothesis of Theorem 1.4, by contrast, must be dense:

Proposition 2.2. *Let n be an integer at least 3. If G is a graph with n vertices that, for each positive integer k less than $n/2$, has fewer than k vertices of degree at most k or fewer than $n - k$ vertices of degree at most $n - k - 1$, then G has at least $n^2/8$ edges.*

Proof. Arrange the degree sequence $d_1, d_2, \ldots, d_n$ of G in non-decreasing order and set $k = \lceil n/2 \rceil - 1$. If $d_k \geq k + 1$, then $d_{n-k} \geq k + 1$ as $d_1 \leq d_2 \leq \ldots \leq d_n$. If $d_k \leq k$, then the sequence has at least k terms not exceeding k, and so it must have fewer than $n - k$ terms not exceeding $n - k - 1$, which means that $d_{n-k} \geq n - k$. Since $n - k \geq k + 1$ by definition, we conclude that

$$\textstyle\sum_{i=1}^{n} d_i \ \geq\ \sum_{i=n-k}^{n} d_i \ \geq\ (k+1)d_{n-k} \ \geq\ (k+1)^2 \ \geq\ n^2/4.$$

□

The lower bound in this proposition can be raised to $(3n^2 - 2n - 8)/16$ by more careful analysis [8, page 96]. (In the same article, its authors also prove that a graph with n vertices has complete closure only if it has at least $\lfloor (n+2)^2/8 \rfloor$ edges and that this bound cannot be improved.)

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