# EXTREMAL SET THEORY

## 1 Sperner's theorem

The Erdős-Rado investigations of $\Delta$-systems constitute a part of *extremal set theory*, which concerns extremal sizes of set families with prescribed properties (such as including only $k$-point sets and containing no $\Delta$-system of more than $m$ sets). A classic of this theory comes from Emanuel Sperner (1905 – 1980):

**Theorem 1.1** (Sperner [49]). *Let $n$ be a positive integer. If $V$ is an $n$-point set and $\mathcal{E}$ is a family of subsets of $V$ such that*

$$S, T \in \mathcal{E}, S \neq T \Rightarrow S \not\subseteq T, \tag{1}$$

*then*

$$|\mathcal{E}| \leq \binom{n}{\lfloor n/2 \rfloor}. \tag{2}$$

*Furthermore, (2) holds as equality if and only if $\mathcal{E}$ consists of all subsets of $V$ that have size $\lfloor n/2 \rfloor$ or all subsets of $V$ that have size $\lceil n/2 \rceil$.*

A family $\mathcal{E}$ of sets with property (1) is called an *antichain.*

The original proof of Theorem 1.1 involves an intermediate result of independent interest:

**Lemma 1.1.** *Let $n$ and $k$ be integers such that $1 < k < n$; let $V$ be an $n$-point set, let $\mathcal{F}$ be a nonempty family of $k$-point subsets of $V$, and let $\mathcal{F}^{(+)}$, $\mathcal{F}^{(-)}$ be defined by*

$$\begin{aligned} \mathcal{F}^{(+)} &= \{T \subseteq V : \ |T| = k+1 \text{ and } S \subset T \text{ for some } S \text{ in } \mathcal{F}\}, \\ \mathcal{F}^{(-)} &= \{R \subseteq V : \ |R| = k-1 \text{ and } R \subset S \text{ for some } S \text{ in } \mathcal{F}\}. \end{aligned}$$

*Then*

- *(i) if $k < (n-1)/2$, then $|\mathcal{F}^{(+)}| > |\mathcal{F}|$,*
- *(ii) if $k = (n-1)/2$ and $|\mathcal{F}| < \binom{n}{k}$, then $|\mathcal{F}^{(+)}| > |\mathcal{F}|$,*
- *(iii) if $k > (n+1)/2$, then $|\mathcal{F}^{(-)}| > |\mathcal{F}|$,*
- *(iv) if $k = (n+1)/2$ and $|\mathcal{F}| < \binom{n}{k}$, then $|\mathcal{F}^{(-)}| > |\mathcal{F}|$.*

*Proof.* Let $N$ denote the number of all pairs $(S, T)$ such that $S \in \mathcal{F}$, $S \subset T \subseteq V$, and $|T| = k+1$. Each $T$ in $\mathcal{F}^{(+)}$ appears in at most $k+1$ such pairs, and so

$N \le |\mathcal{F}^{(+)}| \cdot (k+1)$; each $S$ in $\mathcal{F}$ appears in precisely $n-k$ such pairs, and so $N = |\mathcal{F}| \cdot (n-k)$; we conclude that

$$\frac{|\mathcal{F}^{(+)}|}{|\mathcal{F}|} \;\ge\; \frac{N}{(k+1)|\mathcal{F}|} \;\ge\; \frac{n-k}{k+1}\,. \tag{3}$$

Proof of (i): Under the assumption $k < (n-1)/2$, we have $(n-k)/(k+1) > 1$ and the conclusion follows from (3).

Proof of (ii): Here, (3) is transformed into

$$\frac{|\mathcal{F}^{(+)}|}{|\mathcal{F}|} \;>\; \frac{N}{(k+1)|\mathcal{F}|} \;=\; 1:$$

the assumption $k = (n-1)/2$ means that $(n-k)/(k+1) = 1$ and under the assumption $|\mathcal{F}| < \binom{n}{k}$, we will prove that $N < |\mathcal{F}^{(+)}| \cdot (k+1)$. To do this, we will find a $T$ in $\mathcal{F}^{(+)}$ with at least one $k$-point subset outside $\mathcal{F}$. Since $|\mathcal{F}| > 0$, there is a $k$-point subset $S$ of $V$ such that $S \in \mathcal{F}$; since $|\mathcal{F}| < \binom{n}{k}$, there is a $k$-point subset $S'$ of $V$ such that $S' \notin \mathcal{F}$; replacing points in $S - S'$ by points in $S' - S$ one by one, we construct a sequence $S_0, S_1, \ldots, S_t$ of $k$-point subsets $S_i$ of $V$ such that $S_0 = S$, $S_t = S'$, and $|S_{i-1} \cap S_i| = k-1$ for all $i = 1, 2, \ldots, t$. If $i$ is the smallest subscript such that $S_i \notin \mathcal{F}$, then $S_{i-1} \in \mathcal{F}$ and we can take $T = S_i \cup S_{i-1}$.

Proofs of (iii) and (iv): In (i) and (ii), replace $\mathcal{F}$ by $\{V - S : S \in \mathcal{F}\}$. $\square$

*Proof of Theorem 1.1.* Let $n$ be a positive integer, let $V$ be an $n$-point set, and let $\mathcal{E}$ be any largest antichain of subsets of $V$; write

$$k_{\min} = \min\{|S| : S \in \mathcal{E}\}, \quad k_{\max} = \max\{|S| : S \in \mathcal{E}\}$$

and

$$\mathcal{E}_{\min} = \{S \in \mathcal{E} : |S| = k_{\min}\}, \quad \mathcal{E}_{\max} = \{S \in \mathcal{E} : |S| = k_{\max}\}.$$

Since $\mathcal{E}$ is an antichain, $\mathcal{E}_{\min}^{(+)}$ is disjoint from $\mathcal{E}$ and $(\mathcal{E} - \mathcal{E}_{\min}) \cup \mathcal{E}_{\min}^{(+)}$ is an antichain; since $\mathcal{E}$ is a largest antichain of subsets of $V$, it follows that $|\mathcal{E}_{\min}^{(+)}| \le |\mathcal{E}_{\min}|$; now part (i) of Lemma 1.1 guarantees that

(i) $k_{\min} \ge (n-1)/2$

and part (ii) of Lemma 1.1 guarantees that

(ii) if $k_{\min} = (n-1)/2$, then $|\mathcal{E}_{\min}| = \binom{n}{(n-1)/2}$.

Similar arguments show that

(iii) $k_{\max} \le (n+1)/2$

and that

(iv) if $k_{\max} = (n+1)/2$, then $|\mathcal{E}_{\max}| = \binom{n}{(n+1)/2}$.

The conclusion of the theorem follows from (i), (ii), (iii), (iv). $\square$

## 1.1 A simple proof of Sperner's theorem

Independently of each other, Koichi Yamamoto [55], Lev Dmitrievich Meshalkin (1934–2000) [38], and David Lubell [37] found the following inequality:

**Theorem 1.2** (The LYM inequality). *If $\mathcal{E}$ is an antichain of subsets of an $n$-point set, then*

$$\sum_{S\in\mathcal{E}} \frac{1}{\binom{n}{|S|}} \;\le\; 1. \tag{4}$$

*Proof.* Let $V$ be an arbitrary but fixed $n$-point set and let $\mathcal{E}$ be an arbitrary but fixed antichain of subsets of $V$. A *chain of length* $n+1$ is any family $S_0, S_1, \ldots, S_n$ of subsets of $V$ such that $|S_i| = i$ for all $i$ and $S_0 \subset S_1 \subset \ldots \subset S_n$ (in particular, $S_0 = \emptyset$ and $S_n = V$). We will count in two different ways the number $N$ of all pairs $(\mathcal{C}, S)$ such that $\mathcal{C}$ is a chain of length $n+1$ and $S \in \mathcal{E}\cap\mathcal{C}$.

For each $k$-point subset $S$ of $V$, there are precisely $k!$ choices of a sequence $S_0$, $S_1$,…, $S_k$ of subsets of $V$ such that $|S_i| = i$ for all $i$ and $S_0 \subset S_1 \subset \ldots \subset S_k = S$; there are precisely $(n-k)!$ choices of a sequence $S_k, S_{k+1}, \ldots, S_n$ of subsets of $V$ such that $|S_i| = i$ for all $i$ and $S = S_k \subset S_{k+1} \subset \ldots \subset S_n$; it follows that each member $S$ of our $\mathcal{E}$ participates in precisely $|S|!(n-|S|)!$ of our pairs $(\mathcal{C}, S)$, and so

$$N = \sum_{S\in\mathcal{E}} |S|!(n-|S|)!\ .$$

Since each chain contains at most one member of any antichain, each $\mathcal{C}$ participates in at most one of our pairs $(\mathcal{C}, S)$, and so $N$ is at most the number of all chains of length $n+1$:

$$N \le n!\ .$$

Comparing the exact formula for $N$ with this upper bound, we get the inequality

$$\sum_{S\in\mathcal{E}} |S|!(n-|S|)!\ \le n!\ ,$$

which is just another way of writing (4). $\square$

Let us prove Theorem 1.1 along the lines of Theorem 1.2. The first part of Theorem 1.1 — inequality (2) — is a direct consequence of Theorem 1.2: every antichain $\mathcal{E}$ of subsets of an $n$-point set satisfies

$$|\mathcal{E}| \;=\; \sum_{S\in\mathcal{E}} 1 \;\le\; \sum_{S\in\mathcal{E}} \frac{\binom{n}{\lfloor n/2\rfloor}}{\binom{n}{|S|}} \;=\; \binom{n}{\lfloor n/2\rfloor} \sum_{S\in\mathcal{E}} \frac{1}{\binom{n}{|S|}} \;\le\; \binom{n}{\lfloor n/2\rfloor}. \tag{5}$$

To prove the second part of Theorem 1.1 — characterization of extremal antichains — along these lines, consider an arbitrary antichain $\mathcal{E}$ of subsets of an $n$-point set $V$ that satisfies both inequalities in (5) with the sign of equality. The first of these equations implies that $\binom{n}{|S|} = \binom{n}{\lfloor n/2\rfloor}$ for all $S$ in $\mathcal{E}$, which

means that $|S| = n/2$ when $n$ is even and $|S| = (n \pm 1)/2$ when $n$ is odd. If $n$ is even, then we are done; if $n$ is odd, then we will argue about the second inequality-turned-equation in (5). This equality means equality in (4); reviewing the proof of Theorem 1.2, we find that
• every chain of length $n+1$ contains a member of $\mathcal{E}$,
and so (since all sets in $\mathcal{E}$ have size $(n \pm 1)/2$)
• for every two subsets $S, T$ of $V$ such that $|S| = (n-1)/2$, $|T| = (n+1)/2$, and $S \subset T$, precisely one of $S$ and $T$ belongs to $\mathcal{E}$.
It follows at once that
• $|S| = |S'| = (n-1)/2,\ \ S \in \mathcal{E},\ S' \subset V,\ \ |S \cup S'| = (n+1)/2 \quad \Rightarrow \quad S' \in \mathcal{E}$
(consider $T = S \cup S'$) and then induction on $|S \cup S'|$ shows that
• $|S| = |S'| = (n-1)/2,\ \ S \in \mathcal{E},\ S' \subset V \quad \Rightarrow \quad S' \in \mathcal{E}$.

# 2 The Erdős-Ko-Rado theorem

In 1938, Paul Erdős, Chao Ko, whose name is also transliterated as Ke Zhao (1910–2002), and Richard Rado proved a theorem that they published twenty-three years later [19, Theorem 1]. Its simplified version presented below is known as the *Erdős-Ko-Rado theorem.*

**Theorem 2.1.** *Let $n$ and $k$ be positive integers such that $2k \le n$. If $V$ is an $n$-point set and $\mathcal{E}$ is a family of $k$-point subsets of $V$ such that*

$$S, T \in \mathcal{E} \Rightarrow S \cap T \neq \emptyset, \tag{6}$$

*then*

$$|\mathcal{E}| \le \binom{n-1}{k-1}. \tag{7}$$

A family $\mathcal{E}$ of sets with property (6) is called an *intersecting family.*

The original proof of Theorem 2.1 involves an intermediate result of independent interest:

**Lemma 2.1.** *Let $V$ be a set and let $\mathcal{E}$ be an intersecting family of subsets of $V$. Given two elements $x, y$ of $V$, write*

$$\mathcal{E}^* \ = \ \{S \in \mathcal{E} :\ x \in S,\ y \notin S,\ (S - \{x\}) \cup \{y\} \notin \mathcal{E}\}$$

*and define $f : \mathcal{E} \to 2^V$ by*

$$f(S) = \begin{cases} (S - \{x\}) \cup \{y\} & \textit{if } S \in \mathcal{E}^*, \\ S & \textit{otherwise.} \end{cases}$$

*Then $\{f(S) :\ S \in \mathcal{E}\}$ is an intersecting family of size $|\mathcal{E}|$.*

*Proof.* We will verify that $S, T \in \mathcal{E} \Rightarrow f(S) \cap f(T) \neq \emptyset$ and that $S \neq T \Rightarrow f(S) \neq f(T)$.

Case 1: $S, T \in \mathcal{E} - \mathcal{E}^*$. In this case, $f(S) = S$, $f(T) = T$, and so both implications hold trivially.

Case 2: $S, T \in \mathcal{E}^*$. In this case, we have $f(S) \cap f(T) \neq \emptyset$ since $y \in f(S) \cap f(T)$. In addition, $S = (f(S) - \{y\}) \cup \{x\}$, $T = (f(T) - \{y\}) \cup \{x\}$, and so $f(S) = f(T) \Rightarrow S = T$.

Case 3: $S \in \mathcal{E}^*$, $T \in \mathcal{E} - \mathcal{E}^*$. In this case, $f(S) = (S - \{x\}) \cup \{y\} \notin \mathcal{E}$ and $f(T) = T$; in particular, $f(S) \neq f(T)$ since $f(S) \notin \mathcal{E}$ and $f(T) \in \mathcal{E}$. It remains to prove that $f(S) \cap f(T) \neq \emptyset$. If $y \in T$, then $y \in f(S) \cap f(T)$ and we are done; now we may assume that

$$y \notin T,$$

and so $f(S) \cap f(T) = (S - \{x\}) \cap T$. If $x \notin T$, then $f(S) \cap f(T) = S \cap T$ and we are done again; now we may assume that

$$x \in T.$$

Since $S \in \mathcal{E}^*$, we have

$$x \in S \text{ and } y \notin S.$$

Now

$$f(S) \cap f(T) \;=\; (S - \{x\}) \cap T \;=\; S \cap (T - \{x\}) \;=\; S \cap ((T - \{x\}) \cup \{y\})$$

Since $x \in T$, $y \notin T$, and $T \in \mathcal{E} - \mathcal{E}^*$, we must have $(T - \{x\}) \cup \{y\} \in \mathcal{E}$; since $\mathcal{E}$ is an intersecting family, we conclude that

$$f(S) \cap f(T) \;=\; S \cap ((T - \{x\}) \cup \{y\}) \;\neq\; \emptyset.$$

□

*Proof of Theorem 2.1.* Let $n$ and $k$ be positive integers such that $2k \leq n$ and let $\mathcal{E}$ be any intersecting family of $k$-point subsets of $\{1, 2, \ldots, n\}$. We will use induction on $n$ to show that $|\mathcal{E}| \leq \binom{n-1}{k-1}$. The induction basis, $n = 2$, is trivial; in the induction step, we assume that $n \geq 3$.

If $k = 1$, then we are done at once: $\mathcal{E}$, being an intersecting family of one-point sets, cannot contain two sets. If $2k = n$, then we are done again: in this case, $\mathcal{E}$ includes at most one set from each pair $(S,\ \{1, 2, \ldots, 2k\} - S)$, and so $|\mathcal{E}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$. Now we may assume that

$$k \geq 2 \quad \text{and} \quad 2k \leq n - 1.$$

Let us define the *weight* $w(\mathcal{F})$ of a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ as $\sum_{S \in \mathcal{F}} \sum_{x \in S} x$. We may assume that among all intersecting families $\mathcal{F}$ of $k$-point subsets of $\{1, 2, \ldots, n\}$ such that $|\mathcal{F}| = |\mathcal{E}|$, family $\mathcal{E}$ has the smallest weight. This assumption guarantees that

$$\begin{array}{l}\text{for every two points } x, y \text{ in } \{1, 2, \ldots, n\} \text{ such that } x > y, \text{ we have} \\ S \in \mathcal{E},\ x \in S,\ y \notin S\ \Rightarrow\ (S - \{x\}) \cup \{y\} \in \mathcal{E}:\end{array} \tag{8}$$

since Lemma 2.1 transforms $\mathcal{E}$ into an intersecting family of $k$-point subsets of $\{1, 2, \ldots, n\}$ that has size $|\mathcal{E}|$ and weight $w(\mathcal{E}) - |\mathcal{E}^*| \cdot (x - y)$, minimality of $w(\mathcal{E})$ implies $\mathcal{E}^* = \emptyset$.

Finally, let us set

$$\mathcal{E}_k \ = \ \{S \in \mathcal{E} :\ n \notin S\} \quad \text{and} \quad \mathcal{E}_{k-1} \ = \ \{S - \{n\} :\ S \in \mathcal{E},\, n \in S\}$$

We will complete the proof by showing that $\mathcal{E}_{k-1}$ is an intersecting family: this assertion and the induction hypothesis imply that

$$|\mathcal{E}| \ = \ |\mathcal{E}_k| + |\mathcal{E}_{k-1}| \ \leq \ \tbinom{n-2}{k-1} + \tbinom{n-2}{k-2} \ = \ \tbinom{n-1}{k-1}.$$

To show that $\mathcal{E}_{k-1}$ is an intersecting family, consider arbitrary sets $A, B$ in $\mathcal{E}_{k-1}$ and take any point $y$ in $\{1, 2, \ldots, n-1\} - (A \cup B)$. By definition, $A \cup \{n\}$ and $B \cup \{n\}$ belong to $\mathcal{E}$; in turn, (8) with $x = n$ and $S = B \cup \{n\}$ guarantees that $B \cup \{y\}$ belongs to $\mathcal{E}$; since $\mathcal{E}$ is an intersecting family, we conclude that

$$A \cap B = (A \cup \{n\}) \cap (B \cup \{y\}) \neq \emptyset.$$

□

## 2.1 A simple proof of the Erdős - Ko - Rado theorem

Gyula Katona [28] found a proof of Theorem 2.1 that imitates the proof of Theorem 1.2. We are going to paraphrase a variation on Katona's theme that comes from Chris Godsil and Gordon Royle [25]. Here, the key notion is a certain family of $n$ sets of size $k$ which we call an *$(n, k)$-ring.* This family is defined by a cyclic order on its underlying set $V$: each member of the resulting ring consists of $k$ points of $V$ that are consecutive in the cyclic order. To put it a little more formally, when $v_1, v_2, \ldots, v_n$ is the cyclic order of the elements of $V$, the members $S_1, S_2, \ldots, S_n$ of the ring are defined as

$$S_i \ = \ \{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \tag{9}$$

with subscript arithmetic modulo $n$ (so that $v_{n+1} = v_1$, $v_{n+2} = v_2$, and so on).

**Lemma 2.2.** *If $n$ and $k$ are positive integers such that $2k \leq n$, then every intersecting subfamily of an $(n, k)$-ring consists of at most $k$ sets.*

*Proof.* Let $n$ and $k$ be positive integers such that $2k \leq n$ and let $\mathcal{F}$ be intersecting subfamily of an $(n, k)$-ring; let $S_1, S_2, \ldots, S_n$ be the members of the ring as in (9). If $\mathcal{F} = \emptyset$, then there is nothing to prove; else $\mathcal{F}$ includes at least one member of the ring and symmetry allows us to assume that $\mathcal{F}$ includes $S_k$. Since every $S_i$ in $\mathcal{F}$ has $S_i \cap S_k \neq \emptyset$, its subscript $i$ must be one of $1, 2, \ldots, 2k-1$; since the two sets in each of the $k-1$ pairs

$$(S_1, S_{k+1}),\ (S_2, S_{k+2}),\ \ldots, (S_{k-1}, S_{2k-1})$$

are disjoint, $\mathcal{F}$ includes at most one of these two sets, and so it includes at most $k$ of the $n$ sets $S_i$. $\square$

*Alternative proof of Theorem 2.1.* Given any positive integers $n$ and $k$ such that $2k \leq n$ and given an arbitrary but fixed $n$-point set $V$, let $M$ denote the number of $(n,k)$-rings $\mathcal{R}$ on $V$ and, for each $k$-point subset $S$ of $V$, let $d(S)$ denote the number of $(n,k)$-rings $\mathcal{R}$ on $V$ such that $S \in \mathcal{R}$. By symmetry, $d(S)$ is a constant $d$ dependent on $n$ and $k$ but independent of the choice of $S$; counting in two different ways the pairs $(\mathcal{R}, S)$ such that $\mathcal{R}$ is an $(n,k)$-ring on $V$ and $S \in \mathcal{R}$, we find that

$$Mn = \tbinom{n}{k} d. \tag{10}$$

Next, given an arbitrary but fixed intersecting family $\mathcal{E}$ of $k$-point subsets of $V$, we will count in two different ways the number $N$ of all pairs $(\mathcal{R}, S)$ such that $\mathcal{R}$ on $V$ is an $(n,k)$-ring and $S \in \mathcal{E} \cap \mathcal{R}$: since each $S$ in $\mathcal{E}$ is featured in $d$ such pairs and, by Lemma 2.2, each $\mathcal{R}$ is featured in at most $k$ such pairs, we have

$$|\mathcal{E}| d = N \leq Mk. \tag{11}$$

Together, (11) and (10) imply that $|\mathcal{E}| \leq \frac{M}{d} k = \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}$. $\square$

## 2.2 Extremal families in the Erdős - Ko - Rado theorem

A family $\mathcal{E}$ of sets such that some point belongs to all members of $\mathcal{E}$ is called a *star*.

**Theorem 2.2** (Theorem 7.8.1 in [25]). *Let $n$ and $k$ be positive integers such that $2k < n$. If $V$ is an $n$-point set and $\mathcal{E}$ is an intersecting family of $k$-point subsets of $V$ such that*

$$|\mathcal{E}| = \binom{n-1}{k-1},$$

*then $\mathcal{E}$ is a star.*

The assumption $2k < n$ of Theorem 2.2 cannot be relaxed to the $2k \leq n$ of Theorem 2.1: for example, the family of all $k$-point subsets of $\{1, 2, \ldots, 2k-1\}$ is intersecting, consists of $\binom{2k-1}{k-1}$ sets, and is not a star.

The remainder of the present section is devoted to a proof of Theorem 2.2.

The folowing lemma is to Lemma 2.2 what Theorem 2.2 is to Theorem 2.1:

**Lemma 2.3** (Lemma 7.7.1 in [25])**.** *If $n$ and $k$ are positive integers such that $2k < n$, then every intersecting subfamily of an $(n, k)$-ring that consists of $k$ sets is a star.*

*Proof.* Let $\mathcal{F}$ and $S_1, S_2, \ldots, S_n$ be as in the proof of Lemma 2.2; in particular, $\mathcal{F} \subset \{S_1, S_2, \ldots, S_{2k-1}\}$ and $S_k \in \mathcal{F}$. Assumption $|\mathcal{F}| = k$ implies that

(i) $\mathcal{F}$ includes precisely one of the two sets $S_i, S_{k+i}$
in each of the $k-1$ pairs $(S_1, S_{k+1})$, $(S_2, S_{k+2})$, $\ldots$, $(S_{k-1}, S_{2k-1})$

since the two sets in each of these pairs are disjoint, and so at most one of them can be included in $\mathcal{F}$. In addition,

(ii) $\mathcal{F}$ includes at most one of the two sets $S_i, S_{k+i+1}$
in each of the $k-2$ pairs $(S_1, S_{k+2})$, $(S_2, S_{k+3})$, $\ldots$, $(S_{k-2}, S_{2k-1})$

since the two sets in each of these pairs are disjoint.

If all $k$ sets $S_1, S_2, \ldots, S_k$ belong to $\mathcal{F}$, then $\mathcal{F}$ is a star (point $v_k$ belongs to all its members) and we are done; now we may assume that there is at least one subscript $j$ such that $1 \le j \le k-1$ and $S_j \notin \mathcal{F}$; let $j$ be the largest subscript with these properties. Now $S_{j+1}, S_{j+2}, \ldots, S_k \in \mathcal{F}$; by (i) and (ii), we have

$$S_i \notin \mathcal{F} \;\Rightarrow\; S_{k+i} \in \mathcal{F} \;\Rightarrow\; S_{i-1} \notin \mathcal{F};$$

referring to these implications with $i = j, j-1, \ldots, 1$, we find that $S_{k+j}$, $S_{k+j-1}$, $\ldots$, $S_{k+1} \in \mathcal{F}$. So $\mathcal{F}$ consists of the $k$ sets $S_{j+1}, S_{j+2}, \ldots, S_{j+k}$ and that makes it a star (point $v_{j+k}$ belongs to all its members). $\square$

Another ingredient of the proof of Theorem 2.2 is this:

**Lemma 2.4.** *Let $n$ and $k$ be positive integers such that $2k < n$, let $V$ be an $n$-point set, and let $A, B, X$ be $k$-point subsets of $V$ such that $A, B$ intersect in precisely one point and $X$ does not include this point. Then there is an $(n, k)$-ring $\mathcal{R}$ with the following properties:*

*(i)* $X \in \mathcal{R}$,
*(ii)* *if $\mathcal{F} \subset \mathcal{R}$ and $\mathcal{F}$ is a star of $k$ sets*
*and every member of $\mathcal{F}$ intersects both $A$ and $B$, then $X \notin \mathcal{F}$.*

*Proof.* Let $w$ denote the single point of $A \cap B$ and let us write $A_0 = A - \{w\}$, $B_0 = B - \{w\}$; let us enumerate

| | |
|---|---|
| the elements of $A_0 - X$ | followed by |
| the elements of $X \cap A_0$ | followed by |
| the elements of $X - (A_0 \cup B_0)$ | followed by |

the elements of $X \cap B_0$ followed by
the elements of $B_0 - X$

as $v_1, v_2, \ldots, v_t$. (Since $w$ is missing from this sequence, we have $t \le n-1$; if $X \subseteq A_0 \cup B_0$, then $t = 2(k-1) \le n-3$.) Then let us enumerate the remaining elements of $V$ as $v_{t+1}, v_{t+2}, \ldots, v_n$ in such a way that

$$w = \begin{cases} v_n & \text{if } X \not\subseteq A_0 \cup B_0, \\ v_{n-1} & \text{if } X \subseteq A_0 \cup B_0. \end{cases}$$

Finally, let $\mathcal{R}$ consist of the sets $S_1, S_2, \ldots, S_n$ defined by

$$S_i \;=\; \{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$$

with subscript arithmetic modulo $n$. By this definition, $X \in \mathcal{R}$; to prove that $\mathcal{R}$ has property (ii), note first that a subfamily of $\mathcal{R}$ is a star of $k$ sets if and only if it is one of $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ defined by

$$\mathcal{F}_i \;=\; \{S_i, S_{i+1}, \ldots, S_{i+k-1}\}$$

with subscript arithmetic modulo $n$. Proving (ii) means proving that

*if an $\mathcal{F}_i$ includes $X$, then it includes a set disjoint from $A$ or $B$ (or both).*

To prove this, we distinguish between two cases.

Case 1: $X \not\subseteq A \cup B$. In this case, $w = v_n$. To begin, note that
- if $i$ is one of $1, \ldots, k$, then $S_k$, disjoint from $A$, is included in $\mathcal{F}_i$;
- if $i$ is one of $k, \ldots, n-k$, then $S_i$, included in $\mathcal{F}_i$, is disjoint from $A$;
- if $i$ is one of $n-k+2, \ldots, n$, then $S_1$, disjoint from $B$, is included in $\mathcal{F}_i$.

To summarize, if $i \ne n-k+1$, then $\mathcal{F}_i$ includes a set disjoint from $A$ or $B$ (or both); we will complete the analysis of this case by showing that $X \cap A = \emptyset$ or else $X \notin \mathcal{F}_{n-k+1}$. To do this, consider the subscript $j$ such that $X = S_j$. If $X \cap A \ne \emptyset$, then (as $v_n \notin X$) we have $1 \le j \le k-1$, and so $S_j \notin \mathcal{F}_{n-k+1}$.

Case 2: $X \subseteq A \cup B$. In this case, $w = v_{n-1}$. To begin, note that
- if $i$ is one of $1, \ldots, k$, then $S_k$, disjoint from $A$, is included in $\mathcal{F}_i$;
- if $i$ is one of $k, \ldots, n-k-1$, then $S_i$, included in $\mathcal{F}_i$, is disjoint from $A$;
- if $i$ is one of $n-k+1, \ldots, n$, then $S_n$, disjoint from $B$, is included in $\mathcal{F}_i$.

To summarize, if $i \ne n-k$, then $\mathcal{F}_i$ includes a set disjoint from $A$ or $B$ (or both); we will complete the analysis of this case by showing that $X \cap A = \emptyset$ or else $X \notin \mathcal{F}_{n-k}$. To do this, consider the subscript $j$ such that $X = S_j$. If $X \cap A \ne \emptyset$, then (as $v_{n-1} \notin X$) we have $j = n$ or $1 \le j \le k-1$, and so $S_j \notin \mathcal{F}_{n-k}$. $\square$

*Proof of Theorem 2.2.* Let $n, k, V, \mathcal{E}$ satisfy the assumptions of the theorem. The argument used in the alternative proof of Theorem 2.1 shows that each $(n,k)$-ring on $V$ includes precisely $k$ members of $\mathcal{E}$ (else we would have $N < Mk$

in (11) and so $|\mathcal{E}| < \binom{n-1}{k-1}$); this and Lemma 2.3 together imply that
($\star$) if $\mathcal{R}$ is an $(n,k)$-ring on $V$, then $\mathcal{R} \cap \mathcal{E}$ is a star of $k$ sets.
In particular, $\mathcal{E}$ includes distinct sets that intersect in precisely one point; let $A, B$ denote them and let $w$ denote the single point of their intersection.

We will complete the proof of the theorem by showing that $w$ belongs to all members of $\mathcal{E}$: given any $k$-point subset $X$ of $V$ such that $w \notin X$, we will prove that $X \notin \mathcal{E}$. To do this, we may assume that $X \cap A \neq \emptyset$ (else $X \notin \mathcal{E}$ follows at once since $\mathcal{E}$ is an intersecting family and $A \in \mathcal{E}$), and so there is an $(n,k)$-ring $\mathcal{R}$ on $V$ with properties (i) and (ii) of Lemma 2.4. Fact ($\star$) guarantees that $\mathcal{R} \cap \mathcal{E}$ is a star of $k$ sets; since every member of $\mathcal{R} \cap \mathcal{E}$ intersects both $A$ and $B$, it follows from Lemma 2.4 that $X \in \mathcal{R}$ and $X \notin \mathcal{R} \cap \mathcal{E}$. □

# 3 Turán numbers

A *hypergraph* is a set $V$ along with a set $E$ of subsets of $V$. Elements of $V$ are the *vertices* of the hypergraph and members of $E$ are its *hyperedges.* If, for some integer $k$, every hyperedge consists of $k$ vertices, then the hypergraph is said to be *$k$-uniform.*

Erdős's close friend Paul Turán (1910 – 1976) asked [52] for the smallest number of hyperedges in a $k$-uniform hypergraph on $n$ vertices in which every set of $\ell$ vertices contains at least one hyperedge. Today, these numbers are called *Turán numbers* and denoted $T(n,\ell,k)$.

## 3.1 When $k \leq 3$

All Turán numbers $T(n,\ell,2)$ have been computed by Turán [51]. When $k \geq 3$, Turán numbers $T(n,\ell,k)$ are hard to compute. Turán conjectured that

$$T(n,4,3) = \begin{cases} (2s-1)(s-1)s & \text{if } n = 3s, \\ (2s-1)s^2 & \text{if } n = 3s+1, \\ (2s+1)s^2 & \text{if } n = 3s+2 \end{cases}$$

and constructed hypergraphs showing that the left-hand side of this conjectured equation is at most its right-hand side. As time progressed, larger and larger families of such hypergraphs have been constructed by Alexandr Kostochka [34], William Brown [5], and Dmitrii Germanovich Fon-Der-Flaass [22]. Abundance of these examples seems to suggest that the conjecture is difficult. Gyula Katona, Tibor Nemetz, and Miklós Simonovits [29] verified it for $n \leq 10$.

In the same paper [29], these three authors proved that

$$\frac{T(n,\ell,k)}{\binom{n}{k}} \geq \frac{T(n-1,\ell,k)}{\binom{n-1}{k}} \quad \text{whenever } n > \ell \geq k. \tag{12}$$

To verify (12), consider a $k$-uniform hypergraph with $n$ vertices and $T(n,\ell,k)$ hyperedges in which every set of $\ell$ vertices contains at least one hyperedge; let $N$ denote the number of pairs $(H,v)$ such that $H$ is a hyperedge and $v$ is a vertex outside $H$. Since every hyperedge appears in precisely $n-k$ of these pairs, we have

$$N = T(n,\ell,k)(n-k);$$

since every set of $\ell$ vertices that does not include $v$ contains at least one hyperedge that does not include $v$, every vertex appears in at least $T(n-1,\ell,k)$ of our pairs, and so

$$N \geq nT(n-1,\ell,k).$$

We conclude that $T(n,\ell,k)(n-k) \geq nT(n-1,\ell,k)$, which is just another way of writing the inequality in (12).

For every choice of positive integers $\ell$ and $k$ such that $\ell \geq k$, the sequence $T(n,\ell,k)/\binom{n}{k}$ with $n = \ell, \ell+1, \ell+2, \ldots$ is nondecreasing by (12) and bounded from above by 1, and so it tends to a limit; let $t(\ell,k)$ denote the value of this limit. In [43], Gerhard Ringel (1919–2008) constructed 3-uniform hypergraphs showing that

$$t(\ell,3) \leq 4/(\ell-1)^2 \tag{13}$$

for all $\ell$. In his construction, the vertex set is split into into $\ell-1$ parts that are as equally large as possible and then the set of these parts is cyclically ordered; now three vertices form a hyperedge if and only if either they belong to the same part or else two of them belong to the same part and the third one belongs to the part that is next in the cyclic order. (When $\ell$ is odd, (13) also follows from another construction: split the vertex set into $\lfloor(\ell-1)/(k-1)\rfloor$ parts that are as equally large as possible and let $k$ vertices form a hyperedge if and only if they belong to the same part.) Turán's conjecture about $T(n,4,3)$ implies $t(4,3) = 4/9$; in addition, he conjectured that $t(5,3) = 1/4$ (see [14, p. 13]); Erdős [13, p. 30] offered \$500 for the determination of even one $t(\ell,k)$ with $\ell > k \geq 3$.

## 3.2 A lower bound on $T(n,\ell,k)$

For every choice of positive integers $q,r,n$ such that $r \leq q \leq n$, Erdős and Hanani [18] defined

> $\overline{\overline{m}}(q,r,n)$ as the largest number of hyperedges in a $q$-uniform hypergraph on $n$ vertices in which every set of $r$ vertices is contained in at most one hyperedge

and

> $\overline{\overline{M}}(q,r,n)$ as the smallest number of hyperedges in a $q$-uniform hypergraph on $n$ vertices in which every set of $r$ vertices is contained in at least one hyperedge.

They noted that [18, inequality (1)]

$$\overline{\overline{\overline{m}}}(q,r,n) \;\le \frac{\binom{n}{r}}{\binom{q}{r}} \;\le \overline{\overline{\overline{M}}}(q,r,n) \quad \text{whenever } r \le q \le n$$

and they stated that it may be conjectured that [1]

$$\overline{\overline{\overline{m}}}(q,r,n) \;\sim\; \overline{\overline{\overline{M}}}(q,r,n) \;\sim\; \frac{\binom{n}{r}}{\binom{q}{r}} \quad \text{whenever } r \le q.$$

This conjecture remained open for over two decades until Vojtěch Rödl [44] proved it by an ingenious semi-random construction. His method became known as *Rödl nibble* and had a great impact on combinatorics (see [31] and the references in its section 1.2).

The *complement* of a hypergraph with vertex set $V$ and hyperedge set $E$ is the hypergraph with vertex set $V$ and hyperedge set $\{V - B : B \in E\}$. Since

> in a $k$-uniform hypergraph on $n$ vertices,
> every set of $\ell$ vertices contains at least one hyperedge
>
> if and only if
>
> in the complement of this hypergraph,
> every set of $n-\ell$ vertices is contained in at least one hyperedge,

we have

$$T(n,\ell,k) = \overline{\overline{\overline{M}}}(n-k, n-\ell, n).$$

Since

$$\binom{n}{n-\ell}\binom{\ell}{k} = \binom{n}{k}\binom{n-k}{n-\ell}$$

(both sides count the number of pairs $(A,B)$ such that $A \subseteq B \subseteq \{1,2,\ldots,n\}$ and $|A| = k$, $|B| = \ell$), the lower bound on $\overline{\overline{\overline{M}}}(q,r,n)$ shows that

$$T(n,\ell,k) \;\ge\; \frac{\binom{n}{k}}{\binom{\ell}{k}} \quad \text{whenever } n \ge \ell \ge k \tag{14}$$

(which, besides following from (12) by induction on $n$, is also easy to prove directly). Rödl's theorem shows that this bound is asymptotically tight in the sense that

$$T(n, n-r, n-q) \;\sim\; \frac{\binom{n}{n-q}}{\binom{n-r}{n-q}} \quad \text{whenever } q \ge r\ . \tag{15}$$

[1] When $f$ and $g$ are real-valued functions defined on positive integers, we write $f(n) \sim g(n)$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 1$.

## 3.3 Turán numbers and Steiner systems

A *Steiner system* with parameters $(n, q, r)$ such that $n \geq q \geq r$ is a $q$-uniform hypergraph on $n$ vertices in which every set of $r$ vertices is contained in precisely one hyperedge. Deciding whether ot not there exists a Steiner system with a prescribed triple of parameters may be a difficult problem and this problem amounts to computing a Turán number:

**Theorem 3.1.** *We have*

$$T(n, \ell, k) \;=\; \frac{\binom{n}{k}}{\binom{\ell}{k}}. \tag{16}$$

*if and only if there is a Steiner system with parameters* $(n, n-k, n-\ell)$.

*Proof.* Given $n, \ell, k$, let $M$ denote the right-hand side of (16). On the one hand, lower bound (14) makes (16) equivalent to the claim that there exists a $k$-uniform hypergraph with $n$ vertices and $M$ hyperedges in which every set of $\ell$ vertices contains at least one hyperedge. On the other hand, the complement of a Steiner system with parameters $(n, n-k, n-\ell)$ is a $k$-uniform hypergraph with $n$ vertices in which every set of $\ell$ vertices contains precisely one hyperedge.

Now consider an arbitrary $k$-uniform hypergraph $\mathcal{H}$ with $n$ vertices in which every set of $\ell$ vertices contains at least one hyperedge. To complete the proof, we will show that $\mathcal{H}$ has precisely $M$ hyperedges if and only if every set of $\ell$ vertices contains precisely one hyperedge. For this purpose, let $m$ denote the number of hyperedges of $\mathcal{H}$ and let $N$ denote the number of pairs $(A, B)$ such that $A$ is a hyperedge, $B$ is a set of $\ell$ vertices, and $A \subseteq B$; in addition, given a set $B$ of $\ell$ vertices, let $w(B)$ denote the number of hyperedges contained in $B$. In this notation,

$$N = m\binom{n-k}{\ell-k}$$

and

$$N = \sum_B w(B)$$

with $B$ running through all sets of $\ell$ vertices. It follows that

$$m = \frac{\sum_B w(B)}{\binom{n-k}{\ell-k}} = \frac{\binom{n}{\ell} + \sum_B (w(B)-1)}{\binom{n-k}{\ell-k}} = M + \frac{\sum_B (w(B)-1)}{\binom{n-k}{\ell-k}}.$$

Since $w(B)-1 \geq 0$ for all $B$, we conclude that $m = M$ if and only if $w(B)-1 = 0$ for all $B$. $\square$

A Steiner system with parameters $(k^2+k+1, k+1, 2)$ is called a *projective plane of order k;* its vertices are referred to as *points* and its hyperedges are referred to as *lines.* There is a unique projective plane of order 2; its seven points can be labeled as $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $p_6$, $p_7$ in such a way that the seven lines are

$$\begin{gathered}\{p_1,p_2,p_4\},\\ \{p_2,p_3,p_5\},\\ \{p_3,p_4,p_6\},\\ \{p_4,p_5,p_7\},\\ \{p_5,p_6,p_1\},\\ \{p_6,p_7,p_2\},\\ \{p_7,p_1,p_3\}.\end{gathered}$$

This plane is also known as the *Fano plane* [21].

On the one hand, Oswald Veblen (1880–1960) and William H. Bussey (1879–1962) found a way of constructing a projective plane of order $k$ whenever $k$ is a prime or a power of a prime [53] (see also [45, Theorem 4.2 on p. 93]); in particular, there are projective planes of orders $2,3,4,5,7,8,9$. On the other hand, Richard H. Bruck (1914–1991) and Herbert J. Ryser (1923–1985) proved that when $k$ mod 4 is 1 or 2, a projective plane of order $k$ exists only if there are integers $x, y$ such that $k = x^2 + y^2$ (which is the case if and only if every prime congruent to 3 modulo 4 occurs with an even exponent in the prime factorization of $k$); in particular, there is no projective plane of order 6.

It had been thought for a long time that the condition of the Bruck-Ryser Theorem, necessary for the existence of a projective plane of a prescribed order, might be also sufficient; in particular, this would imply that there is a projective plane of order 10. As Conway and Pless [9] put it in 1982,

> a question which has long tantalized mathematics is whether or not a projective plane of order 10 can exist.

Theorem 3.1 shows that this question amounts to asking whether $T(111, 109, 100) = 111$. Between 1957 and 1989, some 100 papers dealt with it. This era came to an end with the announcement [36] by Clement Lam, Larry Thiel, and Stan Swiercz (see also [35]): their computer search revealed that there is no such plane.

Apart from these results, we know nothing about the values of $k$ for which there is a projective plane of order $k$. And this is just the tip of an iceberg: What are the values of $n, q, r$ for which there is a Steiner system with parameters $(n, q, r)$?

**Theorem 3.2.** *There is a Steiner system with parameters* $(n, q, r)$ *only if*

$$\binom{q-i}{r-i} \text{ divides } \binom{n-i}{r-i} \text{ for all } i = 0, 1, \ldots, r-1. \tag{17}$$

*Proof.* We shall prove more: In every Steiner system with parameters $(n, q, r)$, every set of $i$ vertices such that $0 \leq i < r$ is contained in precisely

$$\frac{\binom{n-i}{r-i}}{\binom{q-i}{r-i}}$$

hyperedges. For this purpose, given a set $S$ of $i$ vertices such that $0 \leq i < r$, let $R$ denote the family of all sets of $r$ vertices that contain $S$, let $Q$ denote the family of all hyperedges that contain $S$, and let $N$ denote the number of pairs $(A, B)$ such that $A \in R$, $B \in Q$, $A \subseteq B$. On the one hand, for each $A$ in $R$ there is a unique hyperedge $B$ such that $A \subseteq B$; since $S \subseteq A \subseteq B$, we have $B \in Q$. It follows that

$$N = |R| = \binom{n-i}{r-i}.$$

On the other hand, for each $B$ in $Q$ there are precisely $\binom{q-i}{r-i}$ sets $A$ in $R$ such that $A \subseteq B$; it follows that

$$N = |Q|\binom{q-i}{r-i}.$$

Comparing the two expressions for $N$, we conclude that

$$|Q| = \frac{\binom{n-i}{r-i}}{\binom{q-i}{r-i}}.$$

□

Steiner system with parameters $(n, 3, 2)$ are called *Steiner triple systems.* By Theorem 3.2, they exist only if $n$ mod 6 is 1 or 3. Thomas Penyngton Kirkman (1806–1895) proved [33] that this necessary condition for their existence is also sufficient. Haim Hanani (1912–1991) proved in [26] that the necessary condition of Theorem 3.2 is also sufficient when $(q, r) = (4, 3)$ and added in [27] the cases $(q, r) = (4, 2)$ and $(q, r) = (5, 2)$. However, the necessary condition of Theorem 3.2 are not always sufficient: we have already noted that the Bruck-Ryser theorem implies nonexistence of $S(43, 7, 2)$. A theorem of Peter Keevash [31] subsumes the following special case:

> For every pair of positive integers $q, r$ such that $q \geq r$
> there is a positive integer $n_0(q, r)$ such that
> Steiner systems with parameters $(n, q, r)$
> exist for all $n$ satisfying (17) and $n \geq n_0(q, r)$.

The appearance of this result was a great breakthrough: until then, only finitely many Steiner systems with $r \geq 4$ were known and none of them had $r \geq 6$. For more on Steiner systems, see [8].

### 3.4 An upper bound on $T(n,\ell,k)$

The following upper bound is implicit in Erdős's paper [11]:

**Theorem 3.3.**

$$T(n,\ell,k) \;\le\; 1+\frac{\binom{n}{k}}{\binom{\ell}{k}}\ln\binom{n}{\ell} \quad \textit{whenever } n\ge \ell\ge k.$$

*Proof.* We will follow Erdős's argument in the form presented in [7, pp. 435–436]. To begin, let us prove the inequality

$$T\left(\tbinom{n}{k},\tbinom{n}{k}-T(n,\ell,k)+1,\tbinom{\ell}{k}\right) \;\le\; \tbinom{n}{\ell} \tag{18}$$

by exhibiting an $\binom{\ell}{k}$-uniform hypergraph $\mathcal{H}$ with $\binom{n}{k}$ vertices and $\binom{n}{\ell}$ edges, in which every set of $\binom{n}{k}-T(n,\ell,k)+1$ vertices contains at least one hyperedge. To describe $\mathcal{H}$, let $\binom{S}{i}$ denote the set of all $i$-point subsets of a set $S$. In this notation, the vertex set of $\mathcal{H}$ is $\binom{Y}{k}$ for some $n$-point set $Y$ and the hyperedge set of $\mathcal{H}$ is in a one-to-one correspondence with $\binom{Y}{\ell}$: the hyperedge corresponding to a set $X$ in $\binom{Y}{\ell}$ is is $\binom{X}{k}$. Given any set $A$ of $\binom{n}{k}-T(n,\ell,k)+1$ vertices of $\mathcal{H}$, consider the $k$-uniform hypergraph $\mathcal{H}_0$ with vertex set $Y$ and hyperedge set $\binom{Y}{k}-A$. Since $|\binom{Y}{k}-A|<T(n,\ell,k)$, some set $X$ of $\ell$ vertices of $\mathcal{H}_0$ contains no hyperedge of $\mathcal{H}_0$, which means that $\binom{X}{k}$ is disjoint from $\binom{Y}{k}-A$, and so $\binom{X}{k}\subseteq A$, and so $A$ contains a hyperedge of $\mathcal{H}$. This observation completes the proof of (18).

Inequality (18) is a device for transforming lower bounds on Turán numbers into upper bounds on Turán numbers. In particular, lower bound (14) guarantees that

$$T\left(\tbinom{n}{k},\tbinom{n}{k}-T(n,\ell,k)+1,\tbinom{\ell}{k}\right) \;\ge\; \frac{\binom{\binom{n}{k}}{\binom{\ell}{k}}}{\binom{\binom{n}{k}-T(n,\ell,k)+1}{\binom{\ell}{k}}};$$

comparing this inequality with (18), we find that

$$\binom{n}{\ell} \;\geq\; \frac{\binom{\binom{n}{k}}{\binom{\ell}{k}}}{\binom{\binom{n}{k}-T(n,\ell,k)+1}{\binom{\ell}{k}}}, \qquad \text{and so}$$

$$\binom{n}{\ell} \;\geq\; \left(\frac{\binom{n}{k}}{\binom{n}{k}-T(n,\ell,k)+1}\right)^{\binom{\ell}{k}}, \qquad \text{and so}$$

$$\frac{\binom{n}{k}-T(n,\ell,k)+1}{\binom{n}{k}} \;\geq\; \binom{n}{\ell}^{-1/\binom{\ell}{k}}, \qquad \text{and so}$$

$$T(n,\ell,k) \;\leq\; 1+\binom{n}{k}\left(1-\binom{n}{\ell}^{-1/\binom{\ell}{k}}\right).$$

Since $\ln x \leq x-1$ for all positive $x$, we have

$$1-\binom{n}{\ell}^{-1/\binom{\ell}{k}} \;\leq\; -\ln\left(\binom{n}{\ell}^{-1/\binom{\ell}{k}}\right) \;=\; \frac{1}{\binom{\ell}{k}}\ln\binom{n}{\ell};$$

this observation completes the proof of the theorem. □

More on Turán numbers can be found in [47] and elsewhere.

# 4 More general extremal problems

Given a $k$-uniform hypergraph $F$, let $\mathrm{ex}(F,n)$ denote the largest number of hyperedges in a $k$-uniform hypergraph on $n$ vertices that contains no $F$. The task of evaluating *Turán functions* $\mathrm{ex}(F,n)$ subsumes the task of evaluating Turán numbers: we have $T(n,\ell,k)=\binom{n}{k}-\mathrm{ex}(F,n)$, where $F$ is the $k$-uniform hypergraph with $\ell$ vertices and $\binom{\ell}{k}$ hyperedges.

With $F$ the $k$-uniform hypergraph on $2k$ vertices that has two disjoint hyperedges and only these two hyperedges, the Erdős-Ko-Rado theorem asserts that

$$\mathrm{ex}(F,n) \;=\; \binom{n-1}{k-1} \quad \text{whenever } n \geq 2k.$$

Erdős [12] generalized this: if $F$ is the $k$-uniform hypergraph on $tk$ vertices that has $t$ pairwise disjoint hyperedges and only these $t$ hyperedges, then

$$\mathrm{ex}(F,n) \;=\; \binom{n}{k}-\binom{n-t+1}{k} \text{ whenever } n \text{ is sufficiently large relative to } t \text{ and } k.$$

(For $k=2$, this was proved earlier by Erdős and Gallai [15].) Here, the extremal hypergraphs (meaning hypergraphs with $n$ vertices and $\mathrm{ex}(F,n)$ hyperedges that contain no $F$) are constructed by first specifying a set of $t-1$ vertices and then letting a set of $k$ vertices be a hyperedge if and only if it includes at least one of these $t-1$ vertices.

Problems of determining or at least estimating $\mathrm{ex}(F,n)$ for prescribed uniform hypergraphs $F$ constitute a rapidly developing area, which is rich in results. Here is a small sample:

- When $F$ is the Fano plane, we have
$$\mathrm{ex}(F,n) \;=\; \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$
The extremal hypergraphs are constructed by splitting the vertex set into two parts as equally large as possible (which means that their sizes are $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$) and then letting a set of three vertices be a hyperedge if and only if it has at least vertex in each of the two parts. This was conjectured in 1973 by Vera Sós [48] and proved in 2018 by Louis Bellmann and Christian Reiher [4].

- Peter Keevash and Benny Sudakov [32] determined $\mathrm{ex}(F,n)$ when $F$ is the 'expanded triangle', which means the $2k$-uniform hypergraph with vertex set $V_1 \cup V_2 \cup V_3$ and precisely three hyperedges, $V_1 \cup V_2$, $V_2 \cup V_3$, and $V_3 \cup V_1$, such that $V_1, V_2, V_3$ are pairwise disjoint sets of size $k$. For all $n$ that are sufficiently large with respect to $k$, the extremal hypergraphs are constructed by splitting the vertex set into two parts and then letting a set of $2k$ vertices be a hyperedge if and only if it has an odd number of vertices in each of the two parts. (Maximizing the number of hyperedges requires the right choice of the sizes of the two parts, approximately $(n+\sqrt{3n-4})/2$ and $(n-\sqrt{3n-4})/2$.) This proves a conjecture of Frankl [23].

- When $F$ has vertices $1,2,3,4,5$ and hyperedges $\{1,2,3\},\{1,2,4\},\{3,4,5\}$, we have
$$\mathrm{ex}(F,n) \;=\; \left\lfloor \tfrac{n}{3} \right\rfloor \cdot \left\lfloor \tfrac{n+1}{3} \right\rfloor \cdot \left\lfloor \tfrac{n+2}{3} \right\rfloor \quad \text{whenever } n \geq 3000:$$
this is a theorem of Peter Frankl and Zoltán Füredi [24]. Extremal hypergraphs can be constructed by splitting the vertex set into three parts as equally large as possible (which means that their sizes are $\lfloor n/3 \rfloor$, $\lfloor (n+1)/3 \rfloor$, and $\lfloor (n+2)/3 \rfloor$) and then letting a set of three vertices be a hyperedge if and only if it has a vertex in each of the three parts.

More on $\mathrm{ex}(F,n)$ can be found in [30] and elsewhere.

# 5 Chromatic number of hypergraphs

In 1937, E.W. Miller [39] proposed saying that a family of sets has *property B* if there is a set that has a nonempty intersection with each of its members, but does not contain any of them; he used the letter B in honour of Felix Bernstein (1878–1956), whose work in the early years of the twentieth century involved this notion. Later on, Paul Erdős and András Hajnal (1931–2016) asked in [16, p. 119] for the smallest $m(k)$ such that there exists a $k$-uniform hypergraph without property B; they noted that the family of all $k$-point subsets of a $(2k-1)$-point set provides the upper bound $m(k) \leq \binom{2k-1}{k}$ and that $m(3) = 7$, with the upper bound provided by the Fano plane. (About a decade later, Paul Seymour [46] and Bjarne Toft [50] proved that $m(4) \leq 23$; nearly four decades after that, Patric Östergård [40] found by an exhaustive computer search that $m(4) \leq 23$.)

Erdős [10, 11] proved that

$$2^{k-1} < m(k) < k^2 2^{k+1} \tag{19}$$

and then Erdős and László Lovász [20, p. 610] stated that "it seems likely that $m(k)/2^k \to \infty$". This hunch was confirmed by József Beck [2]: he proved that $m(k) \geq \frac{1}{5} 2^k \lg k$ whenever $k \geq 2^{100}$ and then improved this lower bound in [3]. More recently, Jaikumar Radhakrishnan and Aravind Srinivasan [41, Theorem 2.1] improved Beck's bounds even further, to $m(k) \geq \frac{7}{10} 2^k \sqrt{k/\ln k}$ for all sufficiently large $k$, and then Danila D. Cherkashin and Jakub Kozik [6] found a simpler proof of this stronger bound.

In [11], Erdős wrote "A reasonable guess seems to be that $m(k)$ is of the order $k\, 2^k$"; this conjecture remains open. Radhakrishnan and Srinivasan proved that every $k$-uniform hypergraph without property B and with fewer than $k2^k$ hyperedges must have more than $k^2/4k \ln 2k$ vertices [42, Lemma 2].

Erdős and Hajnal [17] defined the *chromatic number* $\chi(H)$ of a hypergraph $H$ (they used the term 'set-system' instead of 'hypergraph') as the smalest $t$ such that the vertex set of $H$ can be split into $t$ sets, none of which contain a hyperedge. They pointed out [17, p. 61] that $\chi(H) \leq 2$ if and only if the hyperedge set of $H$ has property B. Now let $m(k, s)$ denote the smallest number of hyperedge in a $k$-uniform hypergraph of chromatic number greater than $s$. In this notation, $m(k) = m(k, 2)$; we shall prove Erdős's bounds (19) in this more general context.

**Theorem 5.1.** $s^{k-1} < m(k, s) < \lceil k^2 s^{k+1} \ln s \rceil$

*Proof.* To prove the upper bound on $m(k, s)$, let $\mu(n, k, s)$ stand for the smallest possible number of monochromatic $k$-point subsets of an $n$-point set whose points are coloured by $s$ colours. The combinatorial content of the argument used by Erdős in [11] can be extracted in the claim that

$$\begin{gathered}\text{if } s, n, k, \Delta \text{ are positive integers such that } n \geq s(k-1) \\ \text{and } s^n < T(\tbinom{n}{k}, \tbinom{n}{k} - \Delta,\ \mu(n, k, s)), \text{ then } m(k, s) \leq \Delta.\end{gathered} \tag{20}$$

To prove (20), write $V_0 = \{1, 2, \ldots, n\}$ and let $V_1$ denote the set of all $k$-point subsets of $V_0$. Then consider the $\mu(n, k, s)$-uniform hypergraph $H_1$ with vertex-set $V_1$, whose hyperedges are in a one-to-one correspondence with the $s^n$ colourings of $V_0$ by colours $\{1, 2, \ldots, s\}$: for each of these colourings, the corresponding hyperedge consists of some $\mu(n, k, s)$ of the (possibly many more) sets in $V_1$ that are made monochromatic by the colouring. Now assume that

$$s^n < T(\tbinom{n}{k}, \tbinom{n}{k} - \Delta, \mu(n, k, s)).$$

Under this assumption, $H_1$ has fewer than $T(\binom{n}{k}, \binom{n}{k} - \Delta, \mu(n, k, s))$ hyperedges, and so some set $S$ of $\binom{n}{k} - \Delta$ of its vertices contains none of its hyperedges; this means that $V_1 - S$ meets every hyperedge of $H_1$ in at least one vertex, and so every colouring of $V_0$ by $s$ colours makes at least one set in $V_1 - S$ monochromatic. To summarize, the hypergraph with vertex set $V_0$ and hyperedge set $V_1 - S$ is $k$-uniform, has $\Delta$ hyperedges, and its chromatic number is larger than $s$. This proves (20).

The rest is counting. Its ingredients are

(i) $m(k, s) \;\le\; \left\lceil \dfrac{\binom{n}{k} n \ln s}{\mu(n, k, s)} \right\rceil$ for all $n$ such that $n > s(k - 1)$,

(ii) if $s$ divides $n$ and $n \ge sk$, then $\mu(n, k, s) = s\binom{n/s}{k}$,

(iii) $\dbinom{sk^2}{k} \le s^{k+1} \dbinom{k^2}{k}$;

together, these three ingredients imply that

$$m(k, s) \;\le\; \left\lceil \frac{\dbinom{sk^2}{k} sk^2 \ln s}{\mu(sk^2, k, s)} \right\rceil \;=\; \left\lceil \frac{\dbinom{sk^2}{k} k^2 \ln s}{\dbinom{k^2}{k}} \right\rceil \;\le\; \lceil s^{k+1} k^2 \ln s \rceil.$$

To prove (i), note that $\mu(n, k, s) > 0$ if and only if $n > s(k - 1)$ and invoke the Katona-Nemetz-Simonovits lower bound (14) on Turán numbers. This bound implies that

$$T(\tbinom{n}{k}, \tbinom{n}{k} - \Delta, \mu(n, k, s)) \;\ge\; \frac{\dbinom{\binom{n}{k}}{\mu(n, k, s)}}{\dbinom{\binom{n}{k} - \Delta}{\mu(n, k, s)}} \;\ge\; \left( \frac{\binom{n}{k}}{\binom{n}{k} - \Delta} \right)^{\mu(n,k,s)};$$

substituting $\Delta / \binom{n}{k}$ for $t$ in the strict inequality $1 - t < e^{-t}$ which is valid for all nonzero $t$, we get

$$\left( \frac{\binom{n}{k}}{\binom{n}{k} - \Delta} \right)^{\mu(n,k,s)} \;>\; \exp\left( \frac{\Delta \mu(n, k, s)}{\binom{n}{k}} \right).$$

It follows that

$$\Delta \geq \frac{\binom{n}{k} n \ln s}{\mu(n,k,s)} \;\Rightarrow\; T(\tbinom{n}{k}, \tbinom{n}{k} - \Delta, \mu(n,k,s)) > s^n,$$

and so (20) implies that

$$m(k,s) \;\leq\; \left\lceil \frac{\binom{n}{k} n \ln s}{\mu(n,k,s)} \right\rceil .$$

To prove (ii), note that $\mu(n,k,s)$ is the minimum of $\sum_{r=1}^{s} \binom{d_r}{k}$ over all choices of nonnegative integers $d_1, \ldots d_s$ such that $\sum_{r=1}^{s} d_r = n$. Now consider nonnegative integers $d_1, \ldots d_s$ whose average is an integer. Asuming that not all of these integers are equal, we shall find nonnegative integers $c_1, \ldots c_s$ such that $\sum_{r=1}^{s} c_r = \sum_{r=1}^{s} d_r$ and $\sum_{r=1}^{s} \binom{c_r}{k} < \sum_{r=1}^{s} \binom{d_r}{k}$; of course, this will imply (ii). By assumption, some $d_i$ is smaller than average and some $d_j$ is larger than average; since the average is an integer, it follows that $d_j \geq d_i + 2$. Setting

$$c_r = \begin{cases} d_r + 1 & \text{if } r = i, \\ d_r - 1 & \text{if } r = j, \\ d_r & \text{for all other } r, \end{cases}$$

we get

$$\sum_{r=1}^{s} \binom{d_r}{k} - \sum_{r=1}^{s} \binom{c_r}{k} = \binom{d_i}{k} + \binom{d_j}{k} - \binom{d_i+1}{k} - \binom{d_j-1}{k} = \binom{d_j-1}{k-1} - \binom{d_i}{k-1} > 0.$$

To prove (iii), note that

$$\frac{s^{k+1}\binom{k^2}{k}}{\binom{sk^2}{k}} \;=\; s^{k+1} \prod_{i=0}^{k-1} \frac{k^2 - i}{sk^2 - i} \;=\; s^k \prod_{i=1}^{k-1} \frac{k^2 - i}{sk^2 - i}$$

$$\;=\; s \prod_{i=1}^{k-1} \frac{sk^2 - si}{sk^2 - i} \;=\; s \prod_{i=1}^{k-1} \left(1 - \frac{(s-1)i}{sk^2 - i}\right)$$

and that, since $(1-x)^{k-1} \geq 1 - (k-1)x$ for all nonnegative $x$,

$$s \prod_{i=1}^{k-1} \left(1 - \frac{(s-1)i}{sk^2 - i}\right) \;\geq\; s \left(1 - \frac{(s-1)(k-1)}{sk^2 - k + 1}\right)^{k-1}$$

$$\;\geq\; s \left(1 - \frac{(s-1)(k-1)^2}{sk^2 - k + 1}\right) \;\geq\; 1.$$

□

Noga Alon [1] conjectured that $\lim_{s\to\infty} m(k,s)/s^k$ exists for every $k$.

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