Lecture notes for Discrete mathematics of Paul Erdős - NDMI107

EXTREMAL GRAPH THEORY

1 Turán's theorem

A *complete k-partite graph* is a graph whose vertex-set can be split into k pairwise disjoint parts (not necessarily all of them empty) so that two vertices are adjacent if and only if they belong to different parts.

In 1940, while imprisoned in a labour camp in Hungary, Paul Turán proved a seminal theorem:

Theorem 1.1 (Turán [47]). *Let n, r be integers such that $r \geq 2$. Among all the graphs with n vertices and clique number less than r, the unique graph with the largest number of edges is the complete $(r-1)$-partite graph, whose $r-1$ parts have sizes as nearly equal as possible (meaning that every two of these sizes differ by at most one).*

(Erdős [19] later reported that Turán was informed after he finished his paper that the special case $r = 2$ had been proved in 1907 by W. Mantel and others [43].)

We have defined the Turán number $T(n, r, k)$ as the smallest number of hyperedges in a k-uniform hypergraph on n vertices in which every set of r vertices contains at least one hyperedge and we have defined the Turán function $\operatorname{ex}(F, n)$ as the largest number of hyperedges in a k-uniform hypergraph on n vertices that does not contain hypergraph F. Now we will consider the case of $k = 2$, when F is a graph. To begin, let K_r denote the complete graph with r vertices, so that $T(n, r, 2) = \operatorname{ex}(K_r, n)$. Turán's theorem specifies the value of $\operatorname{ex}(K_r, n)$ in an elegant combinatorial way. This specification can be translated into an arithmetic formula, which some find less elegant:

$$\operatorname{ex}(K_r, n) = \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} - \frac{b(r-1-b)}{2(r-1)} \quad \text{where } b = n \bmod (r-1). \tag{1}$$

Let us verify that Theorem 1.1 implies identity (1).

In a graph, adjacent vertices are called *neighbours.*

Given integers r and n such that $2 \leq r \leq n$, consider integers a, b defined by $n = a(r-1) + b$ and $0 \leq b < r-1$. In the complete $(r-1)$-partite graph on n vertices whose $r-1$ parts have sizes as nearly equal as possible, b parts have size $a+1$ and $r-1-b$ parts have size a; consequently, $b(a+1)$ vertices have precisely $n-(a+1)$ neighbours and $(r-1-b)a$ vertices have precisely $n-a$ neighbours. The total number of edges comes to

$$\frac{b(a+1)(n-a-1)+(r-1-b)a(n-a)}{2};$$

it is a routine matter to verify that this quantity equals the right-hand side of (1).

Thirty years later, Erdős found a beautiful refinement of Turán's theorem. There, $d_G(v)$ denotes the *degree* of a vertex v in a graph G, defined as the number of neighbours of v in G.

Theorem 1.2 (Erdős [17]). *Let r be an integer greater than 1. For every graph G with clique number less than r, there is a graph H such that*
(i) G and H share their vertex-set V,
(ii) $d_G(v) \le d_H(v)$ for all v in V,
(iii) H is complete $(r-1)$-partite,
(iv) if $d_G(v) = d_H(v)$ for all v in V, then $H = G$.

To derive Theorem 1.1 from Theorem 1.2, consider any graph G with n vertices and clique number less than r. Theorem 1.2 guarantees the existence of a complete $(r-1)$-partite graph H such that either $H = G$ or else G has fewer edges than H. In particular, if G has the largest number of edges among all the graphs with n vertices and clique number less than r, then G is a complete $(r-1)$-partite graph. Finally, sizes of any two of the $r-1$ parts of G must differ by at most 1: else moving a vertex from the larger part to the smaller one would increase the number of edges in G.

Proof of Theorem 1.2. Given an arbitrary graph G with a nonempty set V of vertices, we shall find

- an integer k greater than 1,
- a clique of $k-1$ vertices in G, and
- a complete $(k-1)$-partite graph H with properties (i), (ii), (iii).

This can be done by the algorithm

$V_1 = V$, $k = 1$;
while $V_k \neq \emptyset$
do choose a vertex w_k in V_k with the largest number of neighbours in V_k;
V_{k+1} = the set of neighbours of w_k in V_k, $k = k+1$;
end
return k and $\{w_1, w_2, \ldots w_{k-1}\}$ and the complete $(k-1)$-partite graph
with parts $V_1 - V_2$, $V_2 - V_3$, ..., $V_{k-1} - V_k$;

(Of course, $V_{k-1} - V_k = V_{k-1}$.) The **while** loop maintains the invariant

vertices $w_1, w_2, \ldots w_{k-1}$ are pairwise adjacent
and adjacent to all vertices in V_k

and so the set $\{w_1, w_2, \dots w_{k-1}\}$ returned by the algorithm is a clique. With H standing for the complete $(k-1)$-partite graph returned by the algorithm, we have

$$d_H(v) = (n - |V_j|) + |V_{j+1}| \;\text{ whenever } v \in V_j - V_{j+1}.$$

In the input graph G, the number of neighbours that any vertex in V_j has in V_j is at most the number of neighbours that w_j has in V_j, which is $|V_{j+1}|$; it follows that

$$d_G(v) \le (n - |V_j|) + |V_{j+1}| \;\text{ whenever } v \in V_j.$$

and so $d_G(v) \le d_H(v)$ for all v in V. Furthermore, if $d_G(v) = d_H(v)$ for a vertex v in $V_j - V_{j+1}$, then v must be adjacent to all the vertices outside V_j; consequently, if $d_G(v) = d_H(v)$ for all v, then

$$u \in V_i - V_{i+1},\, v \in V_j - V_{j+1},\, i < j \;\;\Rightarrow\;\; u \text{ and } v \text{ are adjacent},$$

and so every edge of H is an edge of G; this together with (ii) implies that $G = H$. □

2 The Erdős-Stone theorem

On August 14, 1941, Paul Erdős and two graduate students from Princeton, Shizuo Kakutani (1911 – 2004) and Arthur Stone (1916–2000), were taking a stroll in Southampton on Long Island. When Kakutani took a few photographs of Erdős and Stone against the background of what turned out to be a secret radar station, a guard told them to leave and afterwards reported that "three Japanese had taken pictures of the installation and then departed in a suspicious hurry". The three mathematicians were arrested together at lunch, questioned separately by the FBI, and finally released later that night; the New York *Daily News* reported the incident the next day under the headline 3 ALIENS NABBED AT SHORT-WAWE STATION. Five years later, two of the three aliens published a powerful variation on the theme of Turán's theorem. This variation gives an upper bound on $\mathrm{ex}(K_r(s), n)$, where $K_r(s)$ stands for the complete r-partite graph with precisely s vertices in each part. Its simplified version goes as follows:

Theorem 2.1 (Erdős and Stone [30]). *For every choice of integers r, s and a real number ε such that $r \ge 2,\ s \ge 1,\ \varepsilon > 0$, there is a positive integer $n_0(r, s, \varepsilon)$ such that*

$$n \ge n_0(r, s, \varepsilon) \;\;\Rightarrow\;\; ex(K_r(s), n) < \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2}.$$

To prove Theorem 2.1, we will follow the line of reasoning used by Béla Bollobás and Paul Erdős [6]. The idea is to use induction on r: having found in G a large $K_{r-1}(t)$, we will proceed to find in G a $K_r(s)$ such that, for each $i = 1, 2, \dots, r-1$, the i-th part of the $K_r(s)$ is a subset of the i-th part of the $K_{r-1}(t)$. The heart of the argument goes as follows.

Lemma 2.1. *Let r, s, t be positive integers such that $r \geq 2$ and $s \leq t$. If a graph F contains pairwise disjoint sets $T_1, \ldots, T_r$ of vertices such that*

- *$T_1, T_2, \ldots, T_{r-1}$ are parts of a complete $(r-1)$-partite graph,*
- *$|T_1| = |T_2| = \ldots, |T_{r-1}| = t$ and $|T_r| > (s-1)\binom{t}{s}^{r-1}$,*
- *every vertex in T_r has at least $(r-2)t + s$ neighbours in $\cup_{i=1}^{r-1} T_i$,*

then F contains a $K_r(s)$ with parts $S_1, S_2, \ldots, S_r$ such that $S_i \subseteq T_i$ for all $i = 1, 2, \ldots, r$.

Proof. Since every vertex in T_r has at least $(r-2)t + s$ neighbours in $\cup_{i=1}^{r-1} T_i$, it has at least s neighbours in each of $T_1, T_2, \ldots, T_{r-1}$, and so every vertex w in T_r can be labeled by a tuple $(S_1(w), \ldots, S_{r-1}(w))$ such that each $S_i(w)$ is a set of s neighbours of w in T_i. Since there are $\binom{t}{s}^{r-1}$ possible labels, the lower bound on $|T_r|$ guarantees that at least one label appears on at least s distinct vertices; any s of these vertices may form S_r. □

Proof of Theorem 2.1. Let us write

$$c = 1 - \frac{1}{r-1}.$$

Given r, s, ε, we have to declare a value of $n_0(r, s, \varepsilon)$; then, given any graph G whose number n of vertices is at least $n_0(r, s, \varepsilon)$ and whose number of edges is at least $(c+\varepsilon)\binom{n}{2}$, we have to find a $K_r(s)$ in G. In doing this, we may assume that

$$c + \varepsilon \leq 1;$$

actually, this inequality follows from the lower bound on the number of edges of G.

We will use induction on r. In G, we will find first a complete $(r-1)$-partite graph with parts $T_1, T_2, \ldots, T_{r-1}$ of a large size t and then a set T_r that satisfies the hypothesis of Lemma 2.1. The first step is trivial when $r = 2$ and taken care of by the induction hypothesis when $r > 2$. The second step would be easier to carry out if we could assume that every vertex of G has a large degree. Unfortunately, this is not the case: the assumption that the number of edges of G is at least $(c+\varepsilon)\binom{n}{2}$ means only that the average degree of a vertex in G is at least $(c+\varepsilon)(n-1)$ and allows for individual vertices of very small degrees. Fortunately, we can find in G a subgraph F with a large number m of vertices such that the every vertex of F has degree larger than $(c+\varepsilon/2)(m-1)$ of vertices of F. We will replace G by F at the very start of the proof and carry out both steps in F.

The revised outline goes as follows. Given r, s and ε, we will choose

- first a positive integer t large enough with respect to r, s, ε,
- then a positive integer m_0 large enough with respect to r, s, ε, and t,
- and finally a positive integer n_0 large enough with respect to ε and m_0.

Then we will argue in stages:

STAGE 1: As long as n_0 is large enough with respect to ε and m_0, we can find in G a subgraph F with m vertices such that $m \geq m_0$ and such that every vertex of F has degree larger than $(c + \varepsilon/2)(m - 1)$.

STAGE 2: As long as m_0 is large enough with respect to r and t, we can find in F a complete $(r-1)$-partite graph K with parts of size t.

STAGE 3: As long as t is large enough with respect to r, s, ε and m_0 is large enough with respect to r, s, ε, and t, we can find in F more than $(s-1)\binom{t}{s}^{r-1}$ vertices such that each of them has at least $(r-2)t + s$ neighbours in K.

Now for the details. Stage 1 can be carried out by the following algorithm, where $|F|$ denotes the number of vertices in F:

$F = G$;
while F has a vertex v of degree at most $(c + \varepsilon/2)(|F| - 1)$
do remove v (and all the edges that have v for an endpoint) from F;
end

In the graph F produced by this algorithm, every vertex has degree larger than $(c+\varepsilon/2)(|F|-1)$. However, it may not be immediately obvious that F has any vertices at all: as we are peeling off the deficient vertices one by one, we are making F smaller and smaller like a kitten unravelling a ball of wool. Will F not disintegrate completely and disappear in the end? The following computation shows that the answer is an emphatic "no": the total number of edges the algorithm removes from the input graph G in the process of constructing the output graph F is at most $\sum_{i=1}^{n} (c + \varepsilon/2)(i-1)$, and so F is left with at least $(\varepsilon/2) \cdot \binom{n}{2}$ edges. Writing $m = |F|$, we conclude that

$$\binom{m}{2} \;\geq\; (\varepsilon/2)\cdot\binom{n}{2} \;\geq\; (\varepsilon/2)\cdot\binom{n_0}{2},$$

and so $m \geq m_0$ as long as n_0 is large enough with respect to ε and m_0. (Here, "large enough" means $n_0 \geq 1 + (\sqrt{2/\varepsilon})m_0$.)

In Stage 2, we distinguish between two cases. In case $r = 2$ (the induction basis), insisting on $m_0 \geq t$ is enough to guarantee that F contains a set of t vertices. In case $r > 2$ (the induction step), insisting on

$$m_0 \;\geq\; n_0(r-1, t, 1/(r-1)(r-2)$$

is enough to guarantee that F contains a $K_{r-1}(t)$: to see this, note that F has more than $c\binom{m}{2}$ edges and that

$$c \;=\; \left(1 - \frac{1}{r-2}\right) + \frac{1}{(r-1)(r-2)}.$$

In Stage 3, let L denote the set of vertices of F that lie outside K and have at least $(r-2)t+s$ neighbours in K. To get a lower bound on $|L|$, we will estimate in two different ways the number x of edges of F that have one endpoint in K and the other endpoint outside K. Since every vertex in K has more than $(c+\varepsilon/2)(m-1)$ neighbours in F, it has more than $(c+\varepsilon/2)(m-1)-(|K|-1)$ neighbours outside K, and so

$$x \geq |K| \cdot ((c+\varepsilon/2)(m-1)-(|K|-1)) > |K| \cdot ((c+\varepsilon/2)m-|K|).$$

Since every vertex in L has at most $|K|$ neighbours in K and every vertex outside $K \cup L$ has fewer than $(r-2)t+s$ neighbours in K, we have

$$x \leq |L| \cdot |K| + |F-(K \cup L)|((r-2)t+s) \leq |L| \cdot |K| + m((r-2)t+s).$$

Comparing the upper bound on x with the lower bound gives

$$|L| \cdot |K| + m((r-2)t+s) > |K|((c+\varepsilon/2)m-|K|),$$

and so

$$|L| > ((c+\varepsilon/2)m-|K|) - \frac{m((r-2)t+s)}{|K|} = m\left(\frac{\varepsilon}{2} - \frac{s}{(r-1)t}\right) - (r-1)t;$$

as long as t is large enough to guarantee $s/(r-1)/t < \varepsilon/8$ and and m is large enough to guarantee that $m\varepsilon/8 > (r-1)t$, we can conclude that that $|L| > m\varepsilon/4$. To complete the proof, note that $m\varepsilon/4 > (s-1)\binom{t}{s}^{r-1}$ as long as m is large enough with respect to r, s, ε, and t. $\square$

Let $s(r,\varepsilon,n)$ stand for the largest nonnegative integer s such that every graph with n vertices and at least

$$\left(1-\frac{1}{r-1}+\varepsilon\right)\binom{n}{2}$$

edges contains a $K_r(s)$. With this notation, the Erdős-Stone theorem shows that

$$s(r,\varepsilon,n) \to \infty \;\text{ as }\; n \to \infty.$$

The best known bounds on $s(r,\varepsilon,n)$ are

$$(1-\delta)\frac{\log n}{\log(1/\varepsilon)} \;<\; s(r,\varepsilon,n) \;<\; (2+\delta)\frac{\log n}{\log(1/\varepsilon)}$$

(see [38]); for every positive δ, they hold whenever ε is small enough with respect to δ, r and n is large enough with respect to δ, r, ε.

3 The Erdős-Simonovits formula

The *chromatic number* $\chi(F)$ of a graph F is the smallest number of colours that can be assigned to the vertices of F in such a way that every two adjacent vertices receive distinct colours. (This definition is consistent with the definition of the chromatic number of a hypergraph.) Equivalently, $\chi(F)$ is the smallest r such that F is a subgraph of some complete r-partite graph. Graphs F with $\chi(F) \leq 2$ are called *bipartite*.

Paul Erdős and Miklós Simonovits pointed out a fundamental corollary of the Erdős-Stone theorem:

Theorem 3.1 (Erdős and Simonovits [28]). *For every graph F with at least one edge, we have*

$$\lim_{n\to\infty} \frac{ex(F,n)}{\binom{n}{2}} \;=\; 1 - \frac{1}{\chi(F)-1}. \tag{2}$$

Proof. Writing $r = \chi(F)$, $s = |F|$, we claim that

$$\left(1 - \frac{1}{r-1}\right)\binom{n-r+2}{2} \;\leq\; \mathrm{ex}(F,n) \;\leq\; \mathrm{ex}(K_r(s),n); \tag{3}$$

formula (2) follows from (3) combined with Theorem 2.1. To justify the lower bound on $\mathrm{ex}(F,n)$ in (3), observe that it is a lower bound on the number of edges in the complete $(r-1)$-partite graph with parts of size $\lfloor n/(r-1) \rfloor$ and that no complete $(r-1)$-partite graph has a subgraph isomorphic to F. The upper bound is justified by observing that F is a subgraph of $K_r(s)$. $\square$

When f and g are real-valued functions defined on positive integers, we write $f(n) \sim g(n)$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 1$.

The Erdős-Simonovits formula (2) shows that

$$\mathrm{ex}(F,n) \;\sim\; \left(1 - \frac{1}{\chi(F)-1}\right)\binom{n}{2} \quad \text{whenever } \chi(F) \geq 3.$$

4 When F is bipartite

When f and g are real-valued functions defined on positive integers, we write $f(n) = o(g(n))$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 0$.

When $\chi(F) = 2$, the Erdős-Simonovits formula (2) provides no asymptotic formula for the Turán function $\text{ex}(F, n)$: it shows only that

$$\text{ex}(F, n) \;=\; o(n^2).$$

4.1 An Erdős-Simonovits conjecture

Erdős and Simonovits [16, p. 119] conjectured that a simple asymptotic formula for $\text{ex}(F, n)$ exists even if $\chi(F) = 2$:

Conjecture 4.1. *For every bipartite graph F there are constants c and α such that $1 \le \alpha < 2$ and*

$$ex(F, n) \sim cn^{\alpha}. \tag{4}$$

In [21, page 6], Erdős offered \$500 for a proof or disproof. Conjecture 4.1 is known to hold true for certain special choices of F. Let us elaborate.

It follows from a conjecture of Erdős and Vera Sós that (5) every tree T of order k satisfies

$$\text{ex}(T, n) \sim \frac{k-2}{2} n. \tag{5}$$

More precisely, the Erdős-Sós conjecture [13, p. 30] is:

Conjecture 4.2. *Every tree T of order k satisfies*

$$ex(T, n) \le \frac{k-2}{2} n. \tag{6}$$

Since every tree T of order k satisfies $\text{ex}(T, n) \ge (k-2)n/2$ whenever n is a multiple of k (to see this, consider the disjoint union of complete graphs of order $k-1$), inequality (6) implies (5).

If T is the star of order k, then clearly $\text{ex}(T, n) = \lfloor (k-2)n/2 \rfloor$, which implies (6). An old result of Erdős and Gallai [25, Theorem(2.6)] asserts that the path of order k also satisfies (6) in place of T. For additional classes of trees which satisfy the Erdős-Sós conjecture, see [42, 10, 46, 32]. Miklós Ajtai, János Komlós, and Endre Szemerédi have announced that they have proved (6) for all trees T of order k and all n sufficiently large with respect to k.

Another class of graphs known to satisfy the Erdős-Simonovits conjecture are particular complete bipartite graphs: Zoltán Füredi [35] proved that

$$\mathrm{ex}(K_{r,2}, n) \;\sim\; \frac{\sqrt{r-1}}{2} n^{3/2} \quad \text{whenever } r \geq 2.$$

(The special case of $r = 2$ had been established three decades earlier by William Brown [9] and, independently and simultaneously, by Erdős, Alfréd Rényi, and Vera Sós [27].) In addition,

$$\mathrm{ex}(K_{3,3}, n) \;\sim\; \frac{1}{2} n^{5/3}$$

has been established by Brown's lower bound in [9] and Füredi's matching upper bound in [34].

4.2 A digression: Jensen's inequality and binomial coefficients

A real-valued function f defined on an interval I is called *convex* if

$$x, y \in I,\, 0 \leq \lambda \leq 1 \;\Rightarrow\; f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Straightforward induction on n shows that

$$f(\textstyle\sum_{i=1}^{n} \lambda_i x_i) \;\leq\; \textstyle\sum_{i=1}^{n} \lambda_i f(x_i)$$

for every convex function f, for all numbers $x_1, \ldots, x_n$ in its domain, and for all choices of nonnegative numbers $\lambda_1, \ldots, \lambda_n$ that sum up to 1. This is the finite form of an inequality published in 1906 by Danish engineer Johan Ludwig William Valdemar Jensen (1859–1925), who, while successful in his career at the Copenhagen Telephone Company, worked on mathematics in his spare time [39].

The following corollary of Jensen's inequality involves a generalization of the combinatorial notion $\binom{d}{k}$ to a real-valued function $\binom{x}{k}$ of a real variable x that reduces to $\binom{d}{k}$ when x assumes a nonnegative integer value d:

$$\tbinom{x}{k} \;=\; x(x-1)\ldots(x-k+1)/k!$$

Lemma 4.1. *If $d_1, \ldots, d_n$ are nonnegative integers and $\sum_{i=1}^{n} d_i \geq n(k-1)$, then*

$$\sum_{i=1}^{n} \binom{d_i}{k} \geq n \binom{\sum_{i=1}^{n} d_i/n}{k}.$$

Proof. Since the derivative of $\binom{x}{k}$ is an increasing function of x in the interval $(k-1, \infty)$, function $\binom{x}{k}$ is convex in this interval; it follows that the function f defined on all reals by

$$f(x) = \begin{cases} 0 & \text{when } x \leq k-1, \\ \binom{x}{k} & \text{when } x \geq k-1 \end{cases}$$

is convex and so Jensen's inequality with $\lambda_i = 1/n$ for all i guarantees that

$$\sum_{i=1}^{n} \binom{d_i}{k} \;=\; \sum_{i=1}^{n} f(d_i) \;\geq\; nf\left(\sum_{i=1}^{n} d_i/n\right) \;=\; n\binom{\sum_{i=1}^{n} d_i/n}{k}.$$

□

4.3 When F is a complete bipartite graph

Turán' wrote his seminal paper [47] in Hungarian. Thirteen years later, he reproduced the theorem and its proof in a paper [48] written in English and there he put them in a broader context. At the same time, he co-authored with Tamás Kővári and Vera Sós another classic:

Theorem 4.1 (Kővári, Sós, and Turán [40])**.**

$$ex(K_{r,s}, n) \leq (r-1)^{1/s} n^{2-1/s} \;+\; (s-1)n/2. \tag{7}$$

Proof. Given a graph G with n vertices and m edges which contains no $K_{r,s}$, we aim to prove that m is at most the right-hand side of (7). For this purpose, we may assume that

$$m > (s-1)n/2 : \tag{8}$$

otherwise we are done. Under this assumption, consider the set $\mathbf{P}$ of all pairs (v, S) such that v is a vertex of G and S is a set of s neighbours of v. Since each v participates in precisely $\binom{d(v)}{s}$ such pairs, we have $|\mathbf{P}| = \sum_v \binom{d(v)}{s}$; since $\sum_v d(v) = 2m$, it follows from (8) and Lemma 4.1 that

$$|\mathbf{P}| \geq n\binom{2m/n}{s}.$$

Since G contains no $K_{r,s}$, each S participates in at most $r-1$ pairs in $\mathbf{P}$, and so

$$|\mathbf{P}| \leq \binom{n}{s}(r-1).$$

Comparing the two bounds on $|\mathbf{P}|$, we find that

$$n\binom{2m/n}{s} \leq \binom{n}{s}(r-1). \tag{9}$$

Since

$$\frac{\binom{2m/n}{s}}{\binom{n}{s}} \;\geq\; \left(\frac{\dfrac{2m}{n} - (s-1)}{n}\right)^{s},$$

inequality (9) implies that that m is at most the right-hand side of (7). □

When f and g are nonnegative real-valued functions defined on positive integers, we write $f(n) = O(g(n))$ to mean that $f(n) \le cg(n)$ for some constant c and all sufficiently large n; we write $f(n) = \Omega(g(n))$ to mean that $f(n) \ge cg(n)$ for some positive constant c and all sufficiently large n.

The special case $s = r$ of the Kővári-Sós-Turán theorem shows that

$$\mathrm{ex}(K_{r,r}, n) \;=\; O(n^{2-1/r}). \tag{10}$$

As we have seen, this upper bound is tight when $r = 2$ and $r = 3$. For larger values of r, the best known lower bound is weaker:

$$\mathrm{ex}(K_{r,r}, n) \;=\; \Omega(n^{2-2/(r+1)}).$$

This is a special case of a general lower bound:

Theorem 4.2 (Erdős and Joel Spencer [29])**.** *If F is a graph with s vertices and t edges such that $t \ge 2$, then*

$$ex(F, n) \;=\; \Omega(n^{2-(s-2)/(t-1)}).$$

Proof. Given positive integers n and m, let $\mathbf{G}$ denote the set of all graphs with vertices $1, 2, \ldots, n$ and with m edges. Let $\mathbf{P}$ denote the set of all pairs (G, H) such that $G \in \mathbf{G}$ and H is a subgraph of G isomorphic to F. The number of one-to-one mappings from the vertex set of F to $\{1, 2, \ldots, n\}$ is $n(n-1)\cdots(n-s+1)$, and so at most n^s graphs with vertices coming from $\{1, 2, \ldots, n\}$ are isomorphic to F. Since each such graph participates in

$$\binom{\binom{n}{2} - t}{m - t}$$

pairs in $\mathbf{P}$, we have

$$|\mathbf{P}| \;\le\; n^s \binom{\binom{n}{2} - t}{m - t}.$$

Since

$$|\mathbf{G}| \;=\; \binom{\binom{n}{2}}{m},$$

it follows that some G in $\mathbf{G}$ contains at most M subgraphs isomorphic to F, where

$$M \;=\; n^s \frac{\binom{\binom{n}{2} - t}{m - t}}{\binom{\binom{n}{2}}{m}} \;=\; n^s \frac{\binom{m}{t}}{\binom{\binom{n}{2}}{t}} \;\le\; n^s \left(\frac{2m}{n^2}\right)^t.$$

If $m = \lfloor \frac{1}{8} n^{2-(s-2)/(t-1)} \rfloor$, then $n^{s-2t}(2m)^{t-1} \le (1/4)^{t-1}$, and so $M \le m/2$. In this case, removing an edge from each subgraph of G isomorphic to F, we get a graph with n vertices and at least $m/2$ edges which contains no subgraph isomorphic to F. $\square$

4.4 When every subgraph of F has a vertex of degree at most r

Erdős [16, p. 120] conjectured that (10) can be generalized:

Conjecture 4.3.

$$ex(F,n) \;=\; O(n^{2-1/r})$$

for every bipartite graph F such that every subgraph of F has a vertex of degree at most r.

In [24, p. 64] he attributed this conjecture as well as the following companion conjecture jointly to Simonovits and himself.

Conjecture 4.4.

$$ex(F,n) \;=\; \Omega(n^{2+\varepsilon-1/r})$$

for every bipartite graph F with minimum degree greater than r.

For a proof or disproof of each of these conjectures he offered \$500. Regarding the special case $r = 2$, he wrote earlier [20, p. 14]

> Simonovits and I asked: Is it true that [$\text{ex}(F,n) = O(n^{3/2})$ for every bipartite graph F such that every subgraph of F has a vertex of degree at most 2]? We now expect that [this] is false, but can prove nothing.

and [22, pp.64–65]:

> I state some of our favourite conjectures with Simonovits [...] Our conjecture (perhaps more modestly it should be called a guess) is that $\text{ex}(F,n) = O(n^{3/2})$ holds if any only if F is bipartite and has no subgraph each vertex of which has degree greater than 2. Unfortunately we could neither prove the necessity nor the sufficiency of this attractive, illuminating (but perhaps misleading) conjecture .

Noga Alon, Michael Krivelevich and Benny Sudakov [2] proved the existence of a positive constant c such that

$$\text{ex}(F,n) \;=\; O(n^{2-c/r})$$

for every bipartite graph F such that every subgraph of F has a vertex of degree at most r.

4.5 When F is a cycle

In [14, p. 33], Erdős wrote

> ...I can also prove that [$\mathrm{ex}(C_{2k}, n) = O(n^{1+1/k})$] ...

and later [18, p. 78] he commented

> I never published a proof of [$\mathrm{ex}(C_{2k}, n) = O(n^{1+1/k})$] since my proof was messy and perhaps even not quite accurate and I lacked the incentive to fix everything up since I never could settle various related sharper conjectures — all these have now been proved by Bondy and Simonovits — their paper will soon appear.

The first published proof of this upper bound does indeed come from Adrian Bondy and Miklós Simonovits [8]; the best currently known upper bound is Oleg Pikhurko's [44]

$$\mathrm{ex}(C_{2k}, n) \le (k-1)n^{1+1/k} + 16(k-1)n.$$

Erdős and Simonovits [36, Conjecture 4.10] conjectured that $n^{1+1/k}$ is the order of magnitude of $\mathrm{ex}(C_{2k}, n)$ for every constant k:

Conjecture 4.5. *$ex(C_{2k}, n) = \Omega(n^{1+1/k})$.*

This conjecture is known to hold true when $k = 2$ (as we have already seen), when $k = 3$, and when $k = 5$ (these last two lower bounds have been established by Clark Benson [3] in a slightly different setting). When k is arbitrary, Theorem 4.2 gives $\mathrm{ex}(C_{2k}, n) = \Omega(n^{1+/(2k-1)})$ and the best currently known lower bound comes from Felix Lazebnik, Vasiliy Ustimenko, and Andrew Woldar [41]:

$$\mathrm{ex}(C_{2k}, n) = \Omega(n^{1+2/(3k-2)}).$$

Rather than excluding cycles of a single prescribed length, one may consider excluding cycles of all lengths up to a prescribed limit. Such considerations lead to a generalization of the notion of $\mathrm{ex}(F, n)$: when $\mathcal{F}$ is a family of graphs, $\mathrm{ex}(\mathcal{F}, n)$ denotes the largest number of edges in a graph on n vertices that contains no member of $\mathcal{F}$. In particular, $\mathrm{ex}(\{C_3, C_4, \ldots, C_\ell\}, n)$ is the largest number of edges in a graph on n vertices where every cycle has length at least $\ell + 1$. Noga Alon, Shlomo Hoory, and Nathan Linial [1] proved that

$$\mathrm{ex}(\{C_3, C_4, \ldots, C_{2k}\}, n) \le \frac{1}{2}n^{1+1/k} + \frac{1}{2}n.$$

More on Turán functions $\mathrm{ex}(F, n)$ where F is a graph and $\mathrm{ex}(\mathcal{F}, n)$ where $\mathcal{F}$ is a family of graphs can be found, for instance, in [33], [7], and [36].

5 Prehistory

Here is an excerpt (with notation changed for the sake of consistency with these notes) from Erdős's paper [19]:

> As is well known, the theory of extremal graphs really started when Turán determined $\mathrm{ex}(K_r, n)$ and raised several problems which showed the way to further progress. In 1935 I needed (the c's will denote positive absolute constants)
>
> $$\mathrm{ex}(C_4, n) < c_1 n^{3/2} \tag{11}$$
>
> for the following number theoretic problem [. . .] I proved (11) without much difficulty [. . .] I asked if (11) is best possible and Miss E. Klein proved
>
> $$\mathrm{ex}(C_4, n) > c_2 n^{3/2}$$
>
> for every $c_2 > 2^{-3/2}$ and $n > n_0(c_2)$. Being struck by a curious blindness and lack of imagination, I did not at that time extend the problem from C_4 to other graphs and thus missed founding an interesting and fruitful new branch of graph theory.

When Erdős reminisced about this episode in his lectures, he liked to add [45, pp. 153–154]:

> Crookes observed that leaving a photosensitive film near a cathode-ray tube causes damage to the film: it becomes exposed. He concluded that nobody should leave films near a cathode-ray tube. Röntgen observed the same phenomenon a few years later and concluded that this can be used for filming the inside of various objects. [. . .] It is not enough to be in the right place at the right time. You should also have an open mind at the right time.

6 Beyond Turán functions

The term "extremal graph theory" denotes a wide area of results and questions where a graph parameter is maximized subject to other parameters being constrained. Here are two examples:

Theorem 6.1 (Corollary of Theorem 1′ in [26]). *Let G be a graph with n vertices and m edges. If k is a positive integer such that $n > 24k$ and $m \geq (2k-1)n - 2k^2 + k + 1$, then G contains k pairwise vertex-disjoint cycles.*

[1] The lower bound on m in this theorem cannot be reduced. To see this, consider the graph with $2k-1$ vertices of degree $n-1$ and $n-2k+1$ vertices of degree $2k-1$.

Conjecture 6.1 (Case $k=2$ of Conjecture 2 in [23]). *Every triangle-free graph of order n can be made bipartite by deletion of at most $n^2/25$ edges.*

The constant 1/25 in this conjecture cannot be reduced. To see this, consider the graph whose vertex-set is the union of pairwise disjoint sets V_1, V_2, V_3, V_4, V_5 of equal size, where a vertex in V_i is adjacent to a a vertex in V_j if and only if $|i-j|$ is 1 or 4. A weaker version of Conjecture 6.1 with the constant 1/25 raised to 1/18 has been proved by Erdős, Ralph Faudree (1940–2015), János Pach, and Joel Spencer [31, Theorem 2].

Several survey articles, book chapters, and entire books are devoted to extremal graph theory. These include [4], [45], [37, Chapter 10], [5, Chapter IV], [4].

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