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# THE DISCIPLINE NUMBER OF A GRAPH

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## 1. Introduction

The *domination number* $\gamma(G)$ of a graph $G$ is the size of a smallest set $D$ of vertices such that every vertex outside $D$ has at least one neighbour in $D$; Fink, Jacobson, Kinch, and Roberts [4] defined the *bondage number* $b(G)$ of a graph $G$ as the least number of edges whose deletion makes $\gamma(G)$ increase. As we are about to point out, computing $b(G)$ amounts to solving an integer linear program.

Define a *whip* in a graph $G$ as any spanning subgraph $F$ of $G$ such that each component of $F$ is a star and $F$ has precisely $\gamma(G)$ components; let $E(G)$ denote the set of edges of $G$ and let $W(G)$ denote the set of all whips in $G$. Obviously, $b(G)$ is the optimal value of the problem

$$\begin{aligned}
&\text{minimize } \sum \{x_e : e \in E(G)\} \\
&\text{subject to } \sum \{x_e : e \in E(F)\} \geq 1 \quad \text{for all } F \text{ in } W(G), \\
&\qquad x_e \geq 0 \quad \text{for all } e \text{ in } E(G). \\
&\qquad x_e = \text{integer} \quad \text{for all } e \text{ in } E(G).
\end{aligned} \tag{1}$$

By the *fractional bondage number* $b^*(G)$ we shall mean the optimal value of the 'linear programming relaxation' of (1),

$$\begin{aligned}
&\text{minimize } \sum \{x_e : e \in E(G)\} \\
&\text{subject to } \sum \{x_e : e \in E(F)\} \geq 1 \quad \text{for all } F \text{ in } W(G), \\
&\qquad x_e \geq 0 \quad \text{for all } e \text{ in } E(G).
\end{aligned} \tag{2}$$

By the duality theorem of linear programming, $b^*(G)$ equals the optimal value of the dual of (2),

$$\begin{aligned} &\text{maximize } \sum \{y_F\colon F \in W(G)\} \\ &\text{subject to } \sum \{y_F\colon e \in E(F)\} \leqslant 1 \quad \text{for all } e \text{ in } E(G), \\ &\qquad\qquad y_F \geqslant 0 \qquad\qquad\qquad\quad \text{for all } F \text{ in } W(G). \end{aligned} \tag{3}$$

Since (3) can be seen as the linear programming relaxation of

$$\begin{aligned} &\text{maximize } \sum \{y_F\colon F \in W(G)\} \\ &\text{subject to } \sum \{y_F\colon e \in E(F)\} \leqslant 1 \quad \text{for all } e \text{ in } E(G), \\ &\qquad\qquad y_F \geqslant 0 \qquad\qquad\qquad\quad \text{for all } F \text{ in } W(G), \\ &\qquad\qquad y_F = \text{integer} \qquad\qquad\quad \text{for all } F \text{ in } W(G), \end{aligned} \tag{4}$$

problems (1) and (4) are in a sense dual. Therefore we refer to the optimal value of (4) as the *discipline number* dis$(G)$ of $G$.

We have

$$1 \leqslant \mathrm{dis}(G) \leqslant b^*(G) \leqslant b(G) \tag{5}$$

for all graphs $G$. Apart from establishing upper bounds on $b(G)$, Fink et al. computed the bondage number of cycles, paths, and complete multipartite graphs and studied the bondage number of trees (several of these results can also be found in Bauer, Harary, Nieminen, and Suffel [1]). The purpose of this paper is to provide ties with analogous results for the fractional bondage number and for the discipline number.

## 2. The fractional bondage number

The principle restraining device of this section goes as follows.

**Theorem 1.** *If $G$ has $n$ vertices and $m$ edges then $b^*(G) \leqslant m/(n-\gamma(G))$.*

**Proof.** Observe that the constraints of (2) are satisfied by $x_e = 1/(n-\gamma(G))$ for all $e$. □

As usual, let $\Delta(G)$ denote the largest degree of a vertex in $G$. Fink et al. conjectured that $b(G) \leqslant \Delta(G)+1$.

**Theorem 2.** $b^*(G) \leqslant \Delta(G)$.

**Proof.** Consider any maximal set $S$ of pairwise nonadjacent vertices: trivially, every vertex outside $S$ has at least one neighbour in $S$ and the number of edges in $G$ is at most the sum of the degrees of all the vertices outside $S$. Hence the desired conclusion follows from Theorem 1: we have $\gamma(G) \leqslant |S|$ and $m \leqslant \Delta(G)(n - |S|)$. □

Let $C_n$ denote the cycle with $n$ vertices. Fink et al. proved that $b(C_n) = 3$ if $n \equiv 1 \mod 3$, and $b(C_n) = 2$ otherwise. Now we shall prove a theorem that includes a formula for $b^*(C_n)$ as a special case. Recall that a graph $G$ is called *edge-transitive* if for every choice of its edges $e_1$, $e_2$ some automorphism of $G$ sends $e_1$ onto $e_2$.

**Theorem 3.** *If $G$ is edge-transitive with $n$ vertices and $m$ edges then $b^*(G) = m/(n - \gamma(G))$.*

**Proof.** Since $G$ is edge-transitive, (2) has an optimal solution with all $x_e$ equal to each other. Hence $b^*(G)$ is the optimal value of the problem

$$\text{minimize } mx \quad \text{subject to } (n - \gamma(G))x \geqslant 1,\ x \geqslant 0. \qquad \square$$

Since $C_n$ is edge-transitive and $\gamma(C_n) = \lceil n/3 \rceil$, Theorem 3 yields $b^*(C_n) = n/\lfloor 2n/3 \rfloor$.

Let $P_n$ denote the path with $n$ vertices. Fink et al. proved that $b(P_n) = 2$ if $n \equiv 1 \mod 3$, and $b(P_n) = 1$ otherwise.

**Theorem 4.** $b^*(P_n) = \frac{3}{2}$ *if* $n \equiv 1 \mod 3$, *and* $b^*(P_n) = 1$ *otherwise.*

**Proof.** We may assume that $n = 3k + 1$, for otherwise the desired conclusion follows from (5). Since $\gamma(P_{3k+1}) = k + 1$, Theorem 1 guarantees that $b^*(P_{3k+1}) \leqslant \frac{3}{2}$; to prove the reversed inequality, we only need exhibit a feasible solution of (3) in which precisely three variables have value $\frac{1}{2}$. To put it differently, we only need to find three whips in $P_{3k+1}$ so that each edge belongs to precisely two of the three whips. For this purpose, label the edges of $P_{3k+1}$ as $e_1, e_2, \ldots, e_m$ in such a way that $e_i$ and $e_{i+1}$ share an endpoint whenever $1 \leqslant i \leqslant m - 1$. Now the $j$th whip arises by deleting all the edges $e_i$ with $i \equiv j \mod 3$. □

In dealing with complete multipartite graphs, we shall distinguish between those having a positive number $k$ of classes of size one and those in which all classes have size at least two. For the first kind, Fink et al. proved that the bondage number equals $\lceil k/2 \rceil$.

**Theorem 5.** *Let $G$ have $n$ vertices and let precisely $k$ vertices of $G$ have degree $n - 1$. If $k = 1$ then $b^*(G) = 1$; if $k \geqslant 2$ then $b^*(G) = k/2$.*

**Proof.** We may assume that $k \geqslant 2$, for otherwise the desired conclusion follows from (5). Setting $x_e = 1/(k-1)$ if both endpoints of $e$ have degree $n-1$, and $x_e = 0$ otherwise, we obtain a feasible solution of (2); hence $b^*(G) \leqslant k/2$. On the other hand, there are precisely $k$ whips; setting $y_F = \frac{1}{2}$ for all of them, we obtain a feasible solution of (3); hence $b^*(G) \geqslant k/2$. □

For complete multipartite graphs $G$ with class sizes $n_1, n_2, \ldots, n_t$ such that $2 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, Fink et al. proved that $b(G) = n - n_t$ unless $n_1 = n_2 = \cdots = n_t = 2$, in which case $b(G) = n - 1$.

**Theorem 6.** *Let $G$ be a complete multipartite graph with $n$ vertices and $m$ edges. If all classes of $G$ have size at least two then $b^*(G) = m/(n-2)$.*

**Proof.** Theorem 1 guarantees that $b^*(G) \leqslant m/(n-2)$; to prove the reversed inequality, we shall exhibit an appropriate feasible solution of (3). For this purpose, let $S_1, S_2, \ldots, S_t$ denote the classes of $G$; write $n_k = |S_k|$. By a *center* of a star, we shall mean a vertex in the star adjacent to all the other vertices in the star (unless the star has precisely two vertices, its center is uniquely determined); by a *pointed whip,* we shall mean a whip with a center distinguished in each of the two components; the pointed whip is of *type* $(i, j)$ if its two centers belong to $S_i$ and $S_j$. Clearly, there are precisely

$$n_i n_j 2^{n-(n_i+n_j)}$$

pointed whips of type $(i, j)$; each edge with one endpoint in $S_i$ and the other endpoint in $S_j$ belongs to precisely

$$(n_i + n_j - 2)2^{n-(n_i+n_j)}$$

pointed whips of type $(i, j)$, to precisely

$$n_k 2^{n-(n_i+n_k)-1}$$

pointed whips of type $(i, k)$ with $k \neq i, j$, and to precisely

$$n_k 2^{n-(n_j+n_k)-1}$$

pointed whips of type $(k, j)$ with $k \neq i, j$. Hence the desired feasible solution of (3) can be obtained by setting first

$$z_H = \frac{1}{n-2} 2^{n_i+n_j-n}$$

for every pointed whip $H$ of type $(i, j)$, and then

$$y_F = \sum z_H$$

with the summation running through all pointed whips $H$ such that $E(H) = E(F)$. □

Fink et al. proved that $b(T) \leqslant 2$ for every tree $T$.

**Theorem 7.** *$b^*(T) \leqslant (n-1)/\lceil n/2 \rceil$ for every tree T with n vertices.*

**Proof.** As S.T. Hedetniemi pointed out to us, Theorem 13.1.3 in Ore's book [5] implies that $\gamma(G) \leqslant \lfloor n/2 \rfloor$ for every graph without isolated vertices; the rest follows from Theorem 1. □

To show that the bound of Theorem 7 cannot be improved (at least not for even values of $n$), consider the tree with vertices $u_i$, $v_i$ $(1 \leqslant i \leqslant k)$ and edges $u_iu_{i+1}$ $(1 \leqslant i \leqslant k-1)$, $u_iv_i$ $(1 \leqslant i \leqslant k)$. We shall refer to any such tree as a *Justine* [7]. (One of the referees pointed out that the same trees have been called *combs* by Fink et al. [3]. However, *combs* is also the name of graphs used by Padberg and Rinaldi [6] in solving a traveling salesman problem. To avoid confusion, we prefer the descriptive and unambiguous term *Justine*.)

**Theorem 8.** *$b^*(T) = 2(n-1)/n$ for the Justine T with n vertices.*

**Proof.** By virtue of Theorem 7, we only need prove that $b^*(T) \geqslant 2(n-1)/n$; to do this we only need exhibit a feasible solution of (3) in which precisely $n-1$ variables have value $2/n$. To put it differently, we only need find whips $F_1, F_2, \ldots, F_{2k-1}$ in a Justine with $2k$ vertices so that each edge belongs to precisely $k$ of these whips. We propose to do so by induction on $k$. The case of $k=1$ is trivial; now assume that appropriate whips $F_1, F_2, \ldots, F_{2k-3}$ have been found in the Justine with $2k-2$ vertices. Without loss of generality, assume that $F_1, F_2, \ldots, F_{k-2}$ do not include the edge $u_{k-2}u_{k-1}$. Next, observe that each of these $k-2$ whips must include the edge $u_{k-1}v_{k-1}$. Extend each $F_i$ with $1 \leqslant i \leqslant k-2$ by adding the edge $u_{k-1}u_k$ and extend each $F_i$ with $k-1 \leqslant i \leqslant 2k-3$ by adding the edge $u_kv_k$. Finally, let $F_{2k-2}$ consist of all $u_iv_i$ with $i$ odd, all $u_iu_{i+1}$ with $i$ even and less than $k$, and $u_{k-1}u_k$. Let $F_{2k-1}$ consist of all $u_iv_i$ with $i$ even, all $u_iu_{i+1}$ with $i$ odd and less than $k$, and $u_{k-1}u_k$. □

## 3. The discipline number

Theorem 3 combined with (5) implies that $\mathrm{dis}(C_n) = 1$ whenever $n \geqslant 5$; Theorem 7 combined with (5) implies that $\mathrm{dis}(T) = 1$ for every tree $T$; in addition, it is easy to see that $\mathrm{dis}(G) = 1$ whenever $\gamma(G) = 1$. However, we are about to show that $\mathrm{dis}(G)$ can be arbitrarily large even when $\gamma(G) = 2$.

**Theorem 9.** *Let G be a complete multipartite graph with no classes of size one, a classes of size two, and b classes of size at least three. If $a + \lfloor b/2 \rfloor \geqslant 3$, then*

$$\mathrm{dis}(G) = a + \lfloor b/2 \rfloor.$$

*If $(a, b) = (0, 4)$, $(0, 5)$ or $(1, 3)$, then $\mathrm{dis}(G) = 3$. If $(a, b) = (0, 3)$, $(1, 1)$, $(1, 2)$, $(2, 0)$ or $(2, 1)$, then $\mathrm{dis}(G) = 2$. If $(a, b) = (0, 2)$, then $\mathrm{dis}(G) = 1$.*

**Proof.** Enumerate the classes of $G$ as $S_1, S_2, \ldots, S_{a+b}$ so that $|S_i| = 2$ whenever $1 \leq i \leq a$ and $|S_i| \geq 3$ whenever $a + 1 \leq i \leq a + b$.

**Claim 1.** $\text{dis}(G) \geq a + \lfloor b/2 \rfloor$.

**Proof of Claim 1.** Write $S_i = \{u_i, v_i\}$ for $i = 1, 2, \ldots, a$ and choose vertices $u_{a+j}$, $v_{a+j}$ with $1 \leq j \leq \lfloor b/2 \rfloor$ so that $u_{a+j} \in S_{a+2j-1}$, $v_{a+j} \in S_{a+2j}$. For every choice of $i$ and $j$ such that $1 \leq i < j \leq a + \lfloor b/2 \rfloor$, set

$$u_iu_j \in F_i, \quad v_iv_j \in F_i, \quad u_iv_j \in F_j, \quad u_jv_i \in F_j.$$

For all the remaining vertices $w$, set $wu_i \in F_i$ if $w$ and $v_i$ belong to the same $S_k$, and $wv_i \in F_i$ otherwise. □

**Claim 2.** *If* $a + b \geq 4$, *then* $\text{dis}(G) \geq 3$.

**Proof of Claim 2.** Choose vertices $u_1, u_2, u_3$ so that $u_i \in S_i$ and choose a vertex $x$ in $S_4$. Set

$$u_1u_2 \in F_1, \quad u_2u_3 \in F_2, \quad u_3u_1 \in F_3, \quad u_1x \in F_2, \quad u_2x \in F_3, \quad u_3x \in F_1.$$

For all the remaining vertices $w$, set $wx \in F_i$ if $w \in S_i$ and $wu_i \in F_i$ otherwise. □

**Claim 3.** *If* $a \geq 1$, *then* $\text{dis}(G) \geq 2$.

**Proof of Claim 3.** Write $S_1 = \{x_1, x_2\}$. For every vertex $w$ outside $S_1$, set $wx_1 \in F_1$ and $wx_2 \in F_2$. □

These three claims guarantee that the values stated in Theorem 9 provide correct lower bounds on $\text{dis}(G)$; now we shall establish the upper bounds. For this purpose, consider arbitrary pairwise edge-disjoint whips $F_1, F_2, \ldots, F_k$ in $G$. For each $i = 1, 2, \ldots, k$, choose vertices $u_i$, $v_i$ that are centers of the two components of $F_i$. Write

$$Q = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}.$$

**Claim 4.** *If* $|Q| = 2k$, *then* $k \leq a + \lfloor b/2 \rfloor$.

**Proof of Claim 4.** Consider the graph $H$ whose set of vertices is $Q$, two vertices being adjacent in $H$ if and only if they are adjacent in some $F_i$. Since no $u_i$ is adjacent to $v_i$ in $H$, all the remaining pairs of vertices must be adjacent in $H$: we have $\binom{2k}{k} - k = k(2k-2)$ and each $F_i$ contributes $2k - 2$ edges to $H$.

Now call an $S_j$ *special* if it includes at least two vertices from $Q$. As we have just observed, each special $S_j$ includes some $u_i$ and $v_i$ and it includes no other vertices from $Q$; since each vertex outside $Q$ is adjacent to at least one of $u_i$ and $v_i$, we must have $|S_j| = 2$. It follows that $|Q| \leq 2a + b$. □

**Claim 5.** *If $k \geqslant 3$, then $|Q| \geqslant 4$.*

**Proof of Claim 5.** Assume the contrary: $k \geqslant 3$ but $|Q| \leqslant 3$. Since $G$ has at least four vertices, some vertex $w$ lies outside $Q$; since $F_1, F_2, \ldots, F_k$ are edge-disjoint, $w$ is adjacent to at least $k$ distinct vertices in $Q$. Hence $|Q| = k = 3$. Now no $S_j$ can include a vertex from $Q$ and a vertex $w$ outside $Q$ ($w$ has to be adjacent to at least three distinct vertices in $Q$); since $|S_j| \geqslant 2$ for all $j$, it follows that $Q = S_j$ for some $j$. Finally, this $S_j$ includes some vertex $w$ distinct from $u_1$ and $v_1$, a contradiction: $w$ must be adjacent to at least one of $u_1$ and $v_1$. □

**Claim 6.** *If $k \geqslant 4$, then $k \leqslant a + \lfloor b/2 \rfloor$.*

**Proof of Claim 6.** By virtue of Claim 4, we only need show that $|Q| = 2k$. For this purpose, assume the contrary: without loss of generality $u_1 = u_2$. Write

$$Q_0 = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}$$

and consider the graph $H_0$ whose set of vertices is $Q_0$, two vertices being adjacent in $H_0$ if and only if they are adjacent in some $F_i$ with $1 \leqslant i \leqslant 4$. Since each $F_i$ with $1 \leqslant i \leqslant 4$ contributes $|Q_0| - 2$ edges to $H_0$, we have

$$4(|Q_0| - 2) \leqslant \binom{|Q_0|}{2};$$

observing that $|Q_0| \leqslant 7$ (since $u_1 = u_2$) and $|Q_0| \geqslant 4$ (by Claim 5), we conclude that $|Q_0| = 7$. Now $H_0$ has twenty edges, which is a contradiction: $\binom{7}{2} = 21$ and no $u_i$ with $2 \leqslant i \leqslant 4$ is adjacent to $v_i$ in $H_0$. □

**Claim 7.** *If $k = 3$, then $k \leqslant a + \lfloor b/2 \rfloor$ or $a + b \geqslant 4$.*

**Proof of Claim 7.** Claim 4 allows us to assume that $|Q| \leqslant 5$; Claim 5 guarantees that $|Q| \geq 4$. Defining $H$ as in the proof of Claim 4, observe that $H$ has $3(|Q| - 2)$ edges. It follows that $H$ (and hence also $G$) contains four pairwise adjacent vertices. □

**Claim 8.** *If $a = 0$ and $b = 2$, then $k \neq 2$.*

**Proof of Claim 8.** Assume the contrary: $k = 2$ but $a = 0$ and $b = 2$. Claim 4 implies that $|Q| \leqslant 3$ and so, without loss of generality, $u_1 = u_2 \in S_1$. Since $S_1$ includes a vertex distinct from both $u_1$, $v_1$ but adjacent to at least one of them, we must have $v_1 \in S_2$; a symmetric argument shows that $v_2 \in S_2$. But then $S_2$ includes a vertex outside $Q$ and adjacent to only one vertex in $Q$, a contradiction. □

This ties down the proof of Theorem 9. □

The reader interested in additional results in a similar vein is directed to [2, Chapter 5].

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