NDMI107 • 2024 • Solutions to homework 2

1. **Formula:** $\mathrm{ex}(F, n) = \lfloor n^2/4 \rfloor$.
Justification: Since K_3 is a subgraph of F, we have $\lfloor n^2/4 \rfloor = \mathrm{ex}(K_3, n) \leq \mathrm{ex}(F, n)$. To prove that $\mathrm{ex}(F, n) \leq \lfloor n^2/4 \rfloor$, consider an arbitrary graph with n vertices and m edges that does not contain F as a subgraph: we shall prove that $m \leq \lfloor n^2/4 \rfloor$. If F contains no triangle, then $m \leq \mathrm{ex}(K_3, n) = \lfloor n^2/4 \rfloor$ and we are done. If F does contain a triangle, then let T denote the set of its three vertices and let k denote the number of edges of G that have neither endpoint in T. Since G contains no F, no edges of G have exactly one endpoint in T, and so $m \leq 3 + k$. To prove that $3 + k \leq \lfloor n^2/4 \rfloor$, let us use induction on n. If $4 \leq n \leq 6$ (induction basis), then observe that $3 + k \leq 3 + \binom{n-3}{2} \leq \lfloor n^2/4 \rfloor$.If $n \geq 7$ (induction step), then observe that $k \leq \lfloor (n-3)^2/4 \rfloor$ by the induction hypothesis and that $3 + \lfloor (n-3)^2/4 \rfloor \leq \lfloor n^2/4 \rfloor$.

2. **Formula:** $\mathrm{ex}(C_n, n) = \binom{n-1}{2} + 1$.
Justification: Since the graph with n vertices that that consists of K_{n-1} and an additional edge contains no C_n, we have $\binom{n-1}{2} + 1 \leq \mathrm{ex}(C_n, n)$. To prove that $\mathrm{ex}(C_n, n) \leq \binom{n-1}{2} + 1$, consider an arbitrary graph with n vertices and m edges that contains no C_n. By Theorem 11.3 in the book, there is an integer k with $1 \leq k < n/2$ such that $m \leq \binom{n-k}{2} + k^2$. It remains to be shown that $\binom{n-k}{2} + k^2 \leq \binom{n-1}{2} + 1$ whenever k is an integer such that $1 \leq k < n/2$. For this purpose, set $f(x) = (n-x)(n-x-1)/2 + x^2$. We aim to prove that $f(k) \leq f(1)$ for all integers k such that $1 \leq k < n/2$. To simplify the requisite computations, let us prove a stronger statement: $f(x) \leq f(1)$ for all real numbers x such that $1 \leq x \leq n/2$. Since $f''(x) = 3/2$ for all x, function f is convex, and so its maximum in an arbitrary interval is attained

at a boundary of this interval. This observation reduces our task to proving that $f(n/2) \leq f(1)$, which reads $n/4 \leq n^2/8 + 1$; obviously, the last inequality holds whenever $n \geq 2$.

$\mathrm{ex}(K_r, n)$
2.
$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$
The left-hand side counts $(k+1)$-point subsets S of $\{1, 2, \ldots n+1\}$ as follows: First choose the largest element $i+1$ of S and then choose the k-point set $S \cap \{1, 2, \ldots i\}$.