Routing games

Setup

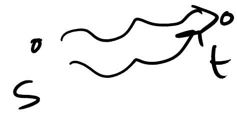
- $G = (V, E)$
- $(s_1, t_1) \dots (s_k, t_k)$ → commodities

  ↳ source sink pairs
- $r_1, \dots r_k$

  ↳ demands

$\mathcal{P}_i$ ... set of paths from s_i to t_i

$\mathcal{P}$ ... $\bigcup_i \mathcal{P}_i$



a flow f is a non-ve vector indexed by $\mathcal{P}$

ie. for each $P \in \mathcal{P}_i$

f_P ... amt of traffic sent from s_i to t_i along P

feasible if

$$\forall i \quad \sum_{P \in \mathcal{P}_i} f_P = r_i$$

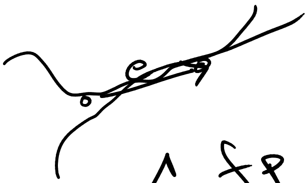
Cost.

Each edge $e \in E$ has a cost function

$c_e : \mathbb{R}^+ \to \mathbb{R}^+$

cts. non-negative non-decreasing

↳ for eg. maps amt of traffic → delay



$\sum_{p:\, e \in P} f_P$

$c_e(0) \neq 0$

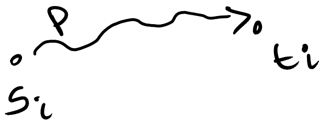
cost of a path

$$c_P(f) = \sum_{e \in P} c_e(f_e)$$

$f_P \geq 0$

f_e

$\sum_{P \in \mathcal{P}:\, e \in P} f_P$



Non-atomic

⇝ each commodity is infinitely divisible

s_i t_i

$r_i = 1$

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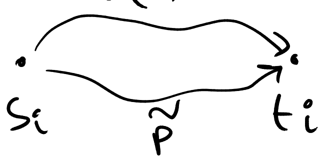
flow is equilibrium iff

$$\forall i \ \forall P, \tilde{P} \in \mathcal{P}_i \ \text{st } f_P > 0$$

$$c_P(f) \leq c_{\tilde{P}}(f)$$

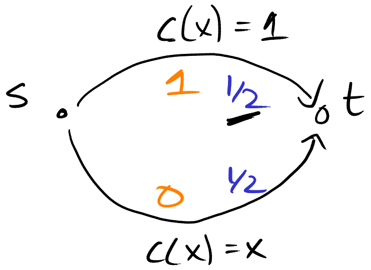
↳ "this is fastest route in current traffic"

$f(P) > 0$



$c_{\tilde{P}}(f) < c_P(f)$

Pigou's example



$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

1

Social cost

$$C(f) = \sum_{p\in\mathcal{P}} c_p(f)\cdot f_p = \sum_{e\in E} c_e(f_e)\cdot f_e$$

$$= \sum_{p\in\mathcal{P}} f_p \sum_{e\in p} c_e(f_e)$$

$$= \sum_{e} c_e(f_e) \underbrace{\sum_{P\in\mathcal{P}:\, e\in P} f_p}_{f_e}$$

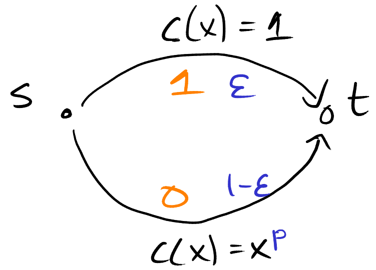
$$C(f) = \sum_{e} C_e(f_e)\, f_e$$

Optimal flow: minimizes $C(f)$

Price of anarchy

$$\frac{\text{worst cost of equilibrium flow}}{\text{cost of optimal flow}}$$

← can use any eq. flow since they all have same cost } non-trivial

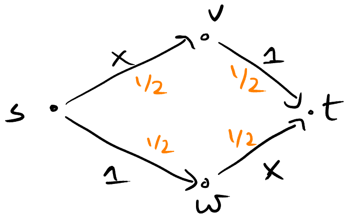


$$\frac{1}{\varepsilon\cdot 1 + (1-\varepsilon)\cdot(1-\varepsilon)^p} \approx p/\ln p$$

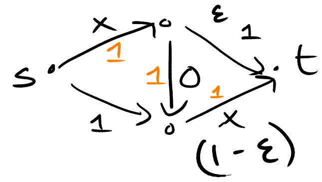
as $p \to \infty$

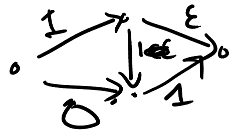
$$1-\varepsilon = \left(\frac{1}{1+p}\right)^{1/p}$$

Braess's Paradox



add edge $\rightsquigarrow$





3

$s \rightsquigarrow t$
$r = 1$

Price of anarchy $\quad 1 \quad \rightsquigarrow \quad 4/3$

$\text{opt} \leq \left(1+\frac{1}{2}\right) = 3/2$

$\text{eq} = 2$

Thm 1 $\rightsquigarrow$ Equilibrium always exists

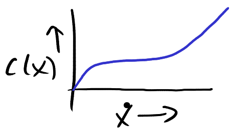
$\rightsquigarrow$ It is "essentially unique".

ie. for $f, \tilde{f}$ eq

$f \quad (G, r, \tilde{c})$ is opt for

$f_P \leq r_i$

$P \in \mathcal{P}_i$



$c_e(f_e) = c_e(\tilde{f}_e)$ for each $e \in E$

Characterizing optimal flows.

$\rightsquigarrow$ suppose $x \cdot c_e(x)$ is convex and continuously differentiable

$$c^*_e(x) = (x \cdot c_e(x))' = c_e(x) + x \cdot c'_e(x)$$

Claim: f is optimal for (G, r, c) $\iff$ f is an equilibrium flow for (G, r, c^*)

Pf. (sketch) [$\Rightarrow$] suppose f is optimal for (G, r, c).

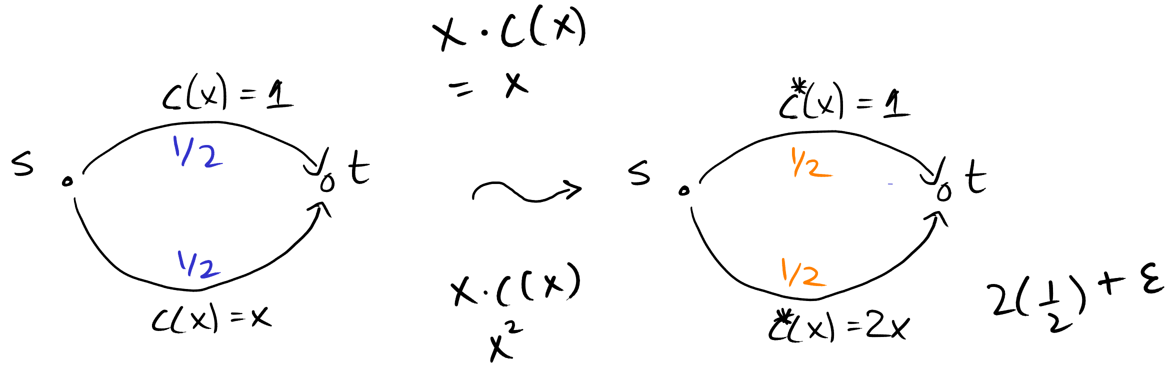
suppose $\exists\, P, \tilde{P} \in \mathcal{P}_i$ s.t. $f_P > 0$, and

$$c^*_P(f) > c^*_{\tilde{P}}(f)$$

imagine $f \rightsquigarrow \tilde{f} = f + \varepsilon \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \to P \\ \to \tilde{P} \\ \\ \\ \end{matrix}$

then $c(\tilde{f}) \approx c(f) + \underbrace{\left(c^*_{\tilde{P}} - c^*_{P}\right)}_{< 0}(f)\cdot\varepsilon$

$\Rightarrow c(\tilde{f}) < c(f)$



equilibrium flows are just opt. flows of a different network

invert claim which used first order derivative.

want $h_e(x)$ s.t. $h_e'(x) = c_e(x)$

def: $h_e(x) = \int_0^x c_e(y)\,dy$

f is an equilibrium flow for (G, r, c) if it is global minima of :

$$\Phi(f) = \sum_{e \in E} \underbrace{\int_0^{f_e} c_e(x)\,dx}_{x\,\tilde{c}(x)}$$

$(G, r, \tilde{c})$

$f, \tilde{f}$

$\tilde{c}(f) = \tilde{c}(\tilde{f})$

$c(f) \overset{?}{=} c(\tilde{f})$

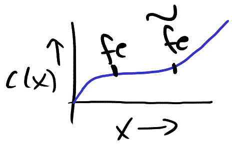
Thm 1 a. Equilibrium always exists

b. It is "essentially unique".

ie. for $f, \tilde{f}$ eq

$$f, \tilde{f}$$

$$C(f) = C(\tilde{f})$$



$$c_e(f_e) = c_e(\tilde{f}_e) \quad \text{for each } e \in E$$

$$f_e = \tilde{f}_e$$

Pf.

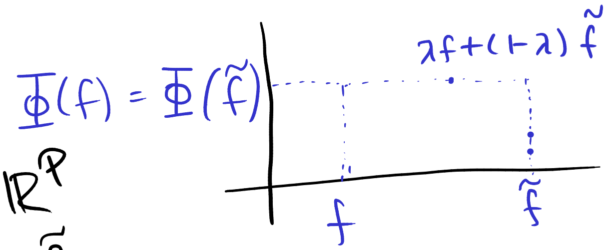
a. • feasible solutions of (G, r, c) form a compact set

• c_e cts $\Rightarrow$ Φ cts

$\Rightarrow$ Φ achieves minimum over set of feasible solutions

b. Φ is cvx, since $c_e(x)$ are non-decreasing

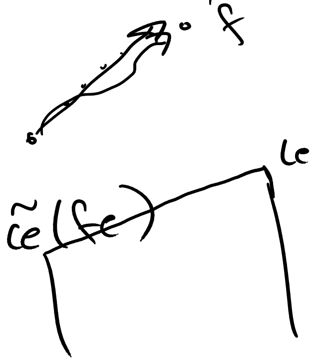
suppose $f, \tilde{f}$ are two different equilibria



$$\Phi(\lambda f + (1-\lambda)\tilde{f}) \leq \rightsquigarrow = \lambda \underbrace{\Phi(f)}_{} + (1-\lambda) \underbrace{\Phi(\tilde{f})}_{}$$

$$\Phi(f) = \Phi(\tilde{f}$$

$f, \tilde{f} \in \mathbb{R}^P$



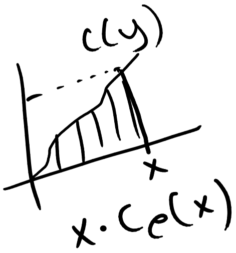
each summand $c_e(\tilde{f}_e) \int_0^x c_e(y)\, dy$ is cvx.

$\Rightarrow$ $\int_0^x c_e(y)\, dy$ linear from f_e to $\tilde{f}_e$.

$\Rightarrow$ $c_e(y)$ const. from f_e to $\tilde{f}_e$.

$$C_e(f_e) = C_e(\tilde{f}_e) \qquad \square$$

Thm 2. (Potential fn upper bound) $\le x \cdot c_e(x)$



(G, r, c) s.t. $x \cdot c_e(x) \le \gamma \cdot \underbrace{\int_0^x c_e(y)\,dy}$

then price of anarchy $\le \gamma$

Pf. Let f, f^* be eq. and opt resp.

$\Phi(f) = \sum_{e \in E} \underbrace{\int_0^{f_e} c_e(x)\,dx}$

$$c(f) \le \gamma \, \underbrace{\Phi}(f)$$
$$\le \gamma \, \Phi\underbrace{(f^*)}$$
$$\le \gamma \, c(f^*) \qquad \square$$

eg. if $c_e(x)$ is polynomial of degree p with non neg. coeff.

$\int_0^x y^p\,dy = \frac{1}{p+1} \cdot x^p$ $\qquad x \cdot c_e(x) \le (p+1) \int_0^x c_e(y)\,dy$ for $x \ge 0$

Thm 2 $\Rightarrow$ price of anarchy $\le p+1$.

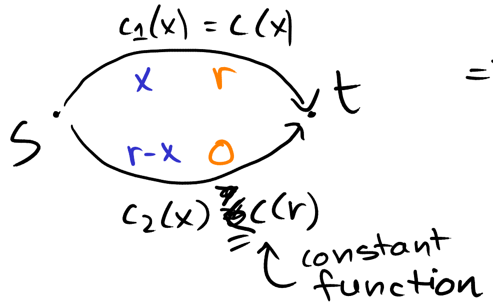
$p / \ln p$

Defn. Pigou bound

for C a set of cost functions + must contain all constants

$$\alpha(C) = \sup_{c \in C} \sup_{x, r \ge 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r-x)\, c(r)}$$

for $c \in C$



$\Rightarrow$ lower bounds price of anarchy for cost functions chosen from C.

r

Prop. (Variational ineq. characterization)

f is eq. flow for (G, r, c) iff

$$\sum_{e \in E} c_e(f_e)\, f_e \leq \sum_{e \in E} c_e(f_e)\, f_e^*$$

for any feasible f^*.

Pf. $$H_f(f^*) = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} c_P(f)\, f_P^* = \sum_{e \in E} c_e(f_e) \cdot f_e^*$$

want: f minimizes H_f

Let f^* be opt for H_f.

$f_P^* > 0 \Rightarrow c_P(f)$ minimum over $P \in \mathcal{P}_i$

f satisfies these conditions iff it is an equilibrium flow.

Thm 3. $\mathcal{C}$ ... set of cost fns. + constant fns.

$\alpha(\mathcal{C})$ ... pigou bound for $\mathcal{C}$

(G, r, c) ... nonatomic instance with $c_e \in \mathcal{C}$ for each $e \in E$.

Then price of anarchy $\leq \alpha(\mathcal{C})$

Pf. Let f, f^* be eq., opt.

$$C(f^*) = \sum_{e \in E} c_e(f_e^*)\, f_e^*$$

$$\alpha(\mathcal{C}) \geq \sup_{\substack{c_e \\ \| \\ c_e}} \sup_{x,r\geq 0} \frac{f_e \cdot c_e(f_e)}{f_e^* c_e(f_e^*) + (f_e - f_e^*) c_e(f_e)}$$

(subst. $x = f_e^*$, $r = f_e$)

$$\Rightarrow \quad f_e^* c_e(f_e^*) \geq \frac{1}{\alpha(\mathcal{C})} f_e c_e(f_e) + (f_e^* - f_e) c_e(f_e)$$

$$\Rightarrow C(f^*) = \sum_{e\in E} \geq \sum_{e \in E}$$

$\sum_e f_e^* c_e(f_e)$

$-\sum_e f_e c_e(f_e) > 0$

$$C(f^*) \geq \frac{1}{\alpha(\mathcal{C})} C(f)$$

$$\frac{C(f)}{\alpha(\mathcal{C}) C^*(f^*)} \leq 1$$

$$\frac{C(f)}{C(f^*)} \leq \alpha(\mathcal{C})$$

Reducing anarchy

Marginal cost pricing

Idea. $c_e \rightsquigarrow c_e^\tau(x) = c_e(x) + \tau_e$

Principle. τ_e should be $f_e \cdot c_e'(f_e)$
for f a feasible flow

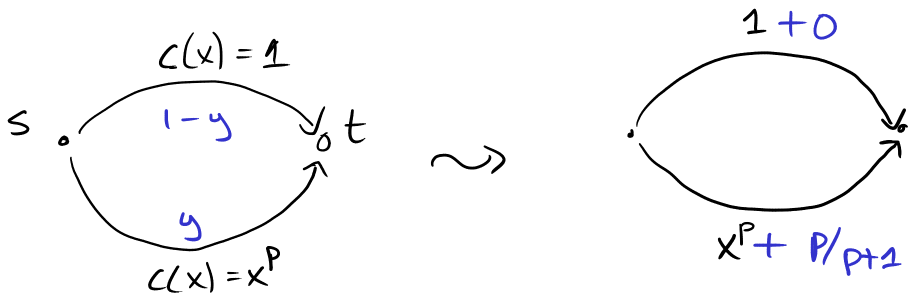
for f^* opt, $\tau_e = f_e^* \cdot c_e'(f_e^*)$
is "missing term" of marginal cost fn.

Thm. f^* opt. for (G, r, c),
let $\tau_e = f_e^* c_e'(f_e^*)$
Then f^* is eq. for $(G, r, c+\tau)$

$$\mathcal{C} = \left\{ \begin{array}{l} \text{constants}, \\ ax + b, \text{ for } a, b > 0 \end{array} \right\}$$

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x, r \geq 0}$$

Eg.



$y = \left(1/1+p\right)^{1/p}$

Thm: If f is an equilibrium flow for (G, r, c) and f^* is feasible for $(G, 2r, c)$ then $C(f) \le C(f^*)$

Pf. Let d_i = min cost path from s_i to t_i under f.

$$C(f) = \sum_{i=1}^{k} d_i r_i$$

Idea. make new cost fns. $\bar{c}_e$ s.t.

i) can lower bound $\bar{C}(f^*)$ in terms of $C(f)$

ii) $\bar{c}_e$ apx c_e

set $\bar{c}_e(x) = \max\{c_e(f_e), c_e(x)\}$

Note that $\bar{C}(f^*) \geq C(f^*)$

while $\bar{C}(f) = C(f)$.

$\forall e$. $\bar{c}_e(x) - c_e(x) \;\; = 0$ for $x \geq f_e$

$\qquad\qquad$ " $\qquad\qquad \le c_e(f_e)$ for $x < f_e$

so. $x\left(\bar{c}_e(x) - c_e(x)\right) \le f_e c_e(f_e)$

for $x \geq 0$.

Thus

$$\bar{C}(f^*) - C(f^*) = \sum_{e \in E} f_e^* \left(\bar{c}_e(f_e^*) - c_e(f_e)\right)$$

$$\leq \sum_{e \in E} f_e \, c_e(f_e) = C(f) \quad -(1)$$

Next, lower bound $\bar{C}(f^*)$

by definition

$$\bar{c}_e(x) \geq c_e(f_e)$$

$\Rightarrow$ min-cost path from s_i to t_i costs $\geq d_i$

Thus $\bar{C}(f^*) = \sum_{P \in \mathcal{P}} \bar{c}_P(f^*) \, f_P^*$

$$\geq \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} d_i f_P^*$$

$$= \sum_{i=1}^{k} 2 r_i d_i = 2C(f) \quad -(2)$$

Combining (1) and (2)

$$\underbrace{\bar{C}(f^*)}_{\geq 2C(f)} - C(f^*) \leq C(f)$$

$$\Rightarrow \quad C(f) \leq C(f^*) \qquad \square$$

Corr. (G, r, c) instance

$\tilde{c}_e(x) := c_e(x/2)/2$

Let $\tilde{f}$ an equilibrium flow for $(G, r, \tilde{c})$, with cost $\tilde{C}(\tilde{f})$

Let f^* be any feasible flow for (G, r, c)

Then $\tilde{C}(\tilde{f}) \leq C(f^*)$

Pf. Double f^* to get feasible flow for $(G, 2r, \cdot)$

Now $\tilde{C}(f) \leq \tilde{C}(2f^*)$

but
$$\tilde{C}(2f^*) = \sum_e 2f_e^* \cdot \tilde{c}_e(2f_e^*)$$
$$= \sum_e f_e^* c_e(f_e^*)$$
$$= C(f^*)$$

Atomic Selfish routing

(G, r, c)

but flows are now "all or nothing"

ie. $f_P^{(i)} = r_i$ for exactly one $P \in \mathcal{P}_i$

and

$f_{\tilde{P}}^{(i)} = 0 \quad \forall \tilde{P} \neq P$

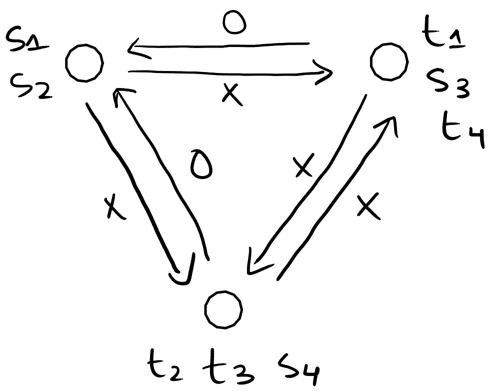
Equilibrium :

f is eq. flow iff

$\forall i \quad \forall P \in \mathcal{P}_i$ s.t. $f_P > 0$,

$$c_P(f) \leq c_{\tilde{P}}(\tilde{f})$$

where
$$\tilde{f} = f + \begin{array}{cc} \begin{pmatrix} 0 \\ \vdots \\ -r_i \\ +r_i \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow P \\ \leftarrow \tilde{P} \\ \\ \\ \end{array} \end{array}$$

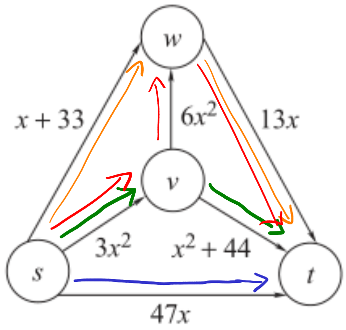


opt.
everyone uses
1-hop path
cost 4

bad eq.
everyone uses
2-hop path
cost 10

price of anarchy 2.5

Equilibrium does not always exist.



P_1 : $s \to t$

P_2 : $s \to v \to t$

P_3 : $s \to w \to t$

P_4 : $s \to v \to w \to t$

two players

$s \rightsquigarrow t$, $r_1 = 1$

$s \rightsquigarrow t$, $r_2 = 2$

1. if player 2 takes P_1 or P_2 player 1 responds P_4

2. if player 2 takes P_3 or P_4 player 1 responds P_1

Thm. Equilibrium exists if all $r_i = R$

Pf. suppose $R = 1$

$$\Phi_a(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

$\Big($ discrete version of $\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)\, dx \Big)$

$\to$ finite # of feasible f

$\Rightarrow$ Φ_a is minimized by some f

claim. f is equilibrium for (G, r, c)

pf. suppose $\exists P, \tilde{P} \in \mathcal{P}_i$

s.t. $c_P(f) > c_{\tilde{P}}(\tilde{f})$

then can reduce Φ_a by switching to $\tilde{f}$.

$$0 < c_p(f) - c_{\tilde{p}}(\tilde{f})$$
$$= \sum_{e \in \tilde{P} \setminus P} c_e(f_e+1) - \sum_{e \in \tilde{P} \setminus P} c_e(f_e)$$

mysterious remark: "best-response dynamics" converge

Thm: Eq. exists if cost functions are affine.

Thm: If (G, r, c) is atomic instance with affine costs,

price of anarchy $\leq \frac{3+\sqrt{5}}{2} \approx 2.618$

if affine + all r_i equal,

price of anarchy $\leq 5/2$ (matches example)