$\boxed{\text{FAIRNESS}}$ based on lectures of $\boxed{\text{A. Igarashi, W. Suksompong}}$

n agents, agent i has a valuation function u_i and receives $M_i \subseteq M$ (resource)

- <u>Envy-freeness</u>: $u_i(M_i) \geq u_i(M_j)$ $\forall i, j$
- <u>Proportionality</u>: $u_i(M_i) \geq \frac{1}{n} u_i(M)$
- <u>Equitability</u>: $u_i(M_i) = u_j(M_j)$

(*) $\boxed{\text{items}}$ < divisible / indivisible] IMPORTANT ELECTIONS

$\boxed{\text{Example}}$ Prime-time presentations of candidates.

Party	
1	5
2	7
3	10
4	5

8 : (2 2 2 2)

12 : (3 3 3 3)
(2 3 5 2)

15 : (3 4 5 3)

MODEL

- $N = \{1, 2, \dots, n\}$ agents
- $M = \{1, \dots, m\}$ indivisible goods (items)
- Allocation : each item goes to at most 1 agent
- Valuation : $(\forall i \leq n)\left(u_i : 2^M \rightarrow \mathbb{Q}\right)$
  - normalised : $u(\emptyset) = 0$
  - monotonic : $u(M_1) \leq u(M_2)$ for $M_1 \subseteq M_2$
  - additive : $u(M') = \sum_{j \in M'} u(j)$

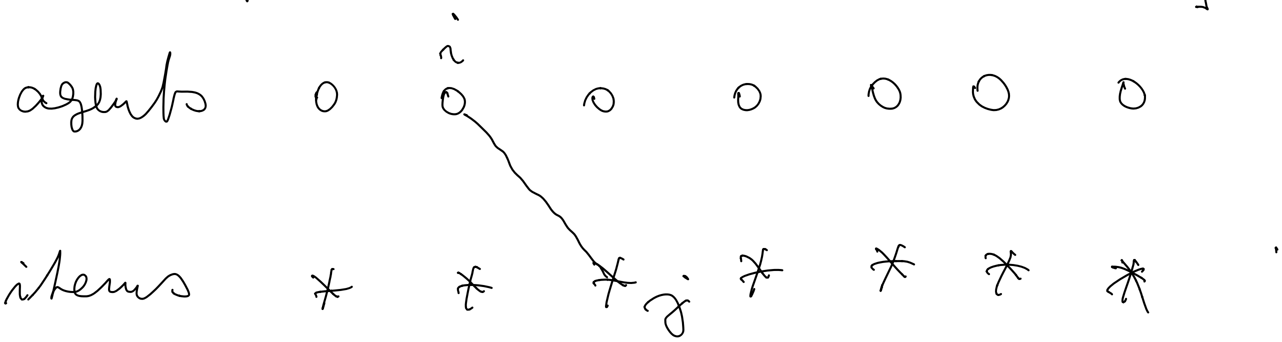
- Utilities additive : $m < n \Rightarrow$ no proportional allocation

Theorem $m = n$, each u_i additive and $u_i(j)$ drawn independently from a distribution. With high probability there is a proportional distribution.

Proportionality

Proof. [for the case that $u_i(j)$ are drawn independently from the uniform distribution over $[0,1]$]

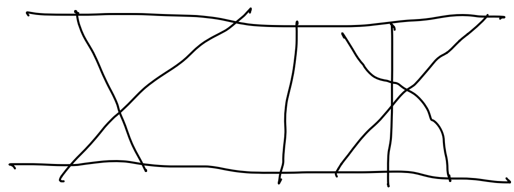
(a)



- add edge $(i\,j)$ if i values j at least $2/3$
- the probability that each edge present is $1/3$
- $\boxed{\text{Theorem (Erdős-Rényi)}}$

[Random graph model] → each edge drawn independently in random with probability $p = p(|V|)$]

$p = \log n/n + \omega(1/n)$ then with high probability, the graph contains a perfect matching.

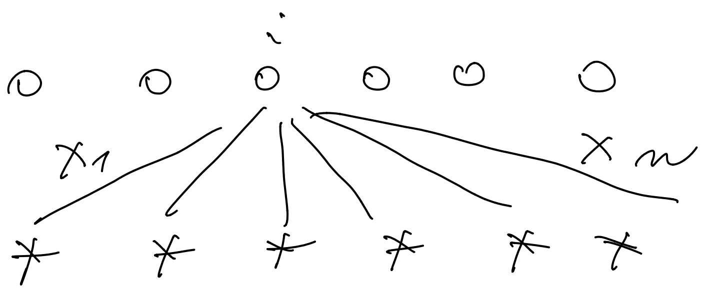


(b) $\boxed{\text{Theorem (Chernoff 52)}}$

$X_1, \dots, X_k$ independent random variables in $[0,1]$ and let $X = X_1 + \dots + X_k$. Then $(\forall \varepsilon \in (0,1))$

$$\Pr\left[X \geq (1+\varepsilon) E[X]\right] \leq e^{-\frac{\varepsilon^2 E[X]}{3}}$$

In our case



$E[X_i] = 1/2$

$E(X) = n/2$

Hence the proportional share of each agent i satisfies

$\frac{1}{n} u_i(M) \sim \frac{1}{2}$ with high probability

$\boxed{\text{Concluding}}$ In the perfect matching, each edge has value $\geq 2/3$ while

$\frac{1}{n} u_i(M) \sim 1/2$.

$\boxtimes$

Envy-Freeness $(\forall_{i,j})(u_i(M_i) \geq u_i(M_j))$

- For additive utilities, envy-freeness implies proportionality:

$$n \cdot u_i(M_i) \geq u_i(M_1) + \dots + u_i(M_n) = u_i(M)$$

- $m < n \Rightarrow$ no envy-free allocation exists
- $m = n$: each agent receives exactly one item $\Rightarrow$ all agents need to have different top items : unlikely!!
- Theorem. $m = \Omega(n \log(n))$ additive

"m behaves like $n \log n$ for large m, n"

Then an allocation maximising social welfare [sum of agents' utilities] is envy-free with high probability.

(*) $m = n + o(n) \Rightarrow$ envy-free allocation unlikely to exist

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Proof again elementary.

additive
Theorem (Manurangsi, Suksompong 2019)

- m divisible by $n \Rightarrow$ envy-free allocation exists with high probability as long as $m \geq 2n$
- else, envy-free allocation is unlikely even when $m = \Theta(n \log n / \log\log n)$.

Envy-freeness up to one good (EF1)

Agent i may envy agent j, but the envy can be eliminated by removing an item from j's bundle.

(*) Can be satisfied by the (additive) round-robin algorithm:

- the agents take turns choosing their favorite item from the remaining items
- i ahead of j then in every round, i does not envy j
- i behind j, then consider the first round

to start with i'th first pick. Then i does not envy j up to j'th first pick.

$\boxed{\text{Nash welfare}}$ of an allocation is the product of the agent's utilities $\prod_{i=1}^{n} u_i(M_i)$

<u>Theorem</u> (Caragiannis et al 2016)

An allocation maximising the $\boxed{\text{Nash welfare}}$ is EF1.

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$\boxed{\text{Pareto optimal}}$ we cannot make some agent better off without making another agent worse off.

$\boxed{\text{Arbitrary monotonic valuations}}$

Lipton et al 2004 EF1

Envy-cycle elimination algorithm

1. Allocate one good at a time in arbitrary order
2. Maintain envy graph with the agents as its vertices, and an edge (i,j) $i \longrightarrow j$ if i envies j with respect to the current partial allocation.
3. At each step, the next good allocated to an agent with <u>no incoming edge</u>.

   Any cycle that arises is eliminated by giving j's bundle to i for any edge $i \longrightarrow j$ in the cycle.

Envy-freeness up to any good (EFX)

i may envy j but the envy eliminated by removing ANY item from j's bundle.

<u>Thm</u> (Plaut, Roughgarden 2018)

An EFX allocation always exists for $\boxed{\text{identical}}$ monotonic valuations.

Arbitrary monotonic valuations: existence open for ≥ 3 agents

K: set of resources; K_i: those available for agent i [Example: different vaccines]

B. Singh, 2020

w_i : weight of agent i

$f_i \in \langle 0,1\rangle$ initial coverage of agent i

$y_i \in \langle 0,1\rangle$ final coverage of agent i

Agent $\to$ a group of people

F : lost function

Allocation is balanced between agents i, i' if

$$w_i F(y_i) = w_{i'} F(y_{i'}) .$$

marginal losses are in inverse proportion to their weights. !!

$\boxed{\text{Justifiable complain}}$

i' has a justifiable complain if

(1) i' has coverage less than 100%

(2) Another user i receives a positive allocation from a resource shared with i'

(3) coverages of i, i' are balanced

(4) i' did not receive any allocation of this shared resource or i received 100% coverage.

Allocation satisfying (1), (2), (3) is $\boxed{\text{proportionally fair}}$ if:

$y_{i'} < 1$ and $x_{ik} > 0$ for $k \in K_i \cap K_{i'}$

$$\Updownarrow$$

$y_i < 1$ and $x_{i'k} > 0$.