Fairness consideration in cooperative games

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Our goals

"What to do, what to do..."

In this presentation: **Fairness**

- revision of already known solution concepts
- introduction to further solution concepts
- an approach to study **fairness** concepts on solution concepts
- an approach to model situations with players with different **fairness** notion

Cooperative game theory

"The cooperation, not the competition, is the main focus here."

Definition

A *cooperative game* is an ordered pair (N, v) where $N = \{1, \ldots, n\}$ is a set of players and $v \colon 2^N \to \mathbb{R}$ is a characteristic function of the cooperative game. We always assume that $v(\emptyset) = 0$.

E.g. $v(\{1, 2, 4\})$ *is the value of cooperation of players 1,2 and 4.*

Solution concepts

"How to split the reward?"

Definition

A payoff vector $x \in \mathbb{R}^n$ represents the profit of ith player as its ith coordinate x_i.

Definition

A payoff vector $x \in \mathbb{R}^n$ is an *imputation* if

- $x_i \geq v(\{i\})$ for $i \in N$ (individual rationality),
- $\sum_{i \in N} x_i = v(N)$ (efficiency).

Solution concepts

"When is the cooperation of everyone a stable situation?"

Definition

A *core* of a game (N, v) is defined as

$$C(v) = \left\{ x \in \mathbb{R}^n | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N \right\}$$

"What is the most *fair* way to distribute the payoffs between players?"

Definition

For a game (N, v) the *Shapley value* for player i is

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S))$$

Solution concepts

"As *close* to $x(S) = v(S)$ as possible..."

- $e(S, x) := v(S) - x(S)$ ... *excess*
- $\theta(x) \in \mathbb{R}^{2^{|N|}}$ ... vector of excesses in non-increasing order

Definition

For a game (N, v), the **nucleolus** $n(v)$ is the minimal imputation x with respect to the lexicographical ordering of $\theta(x)$ i.e.

$$\theta(x) < \theta(y) \text{ if } \exists k : \forall i < k : \theta_i(x) = \theta_i(y) \text{ and } \theta_k(x) < \theta_k(y).$$

Questions and solution concepts

"It makes sense, but tell me..."

Questions concerning solution concepts:

- When $C(v) \neq \emptyset$? (properties of concepts)
- If $|C(v)| \geq 2$, how to choose $x \in C(v)$?
- $\phi(v) \in C(v)$? (relations between concepts)
- How to compute $C(v)$? (computating the concepts)

- because of general definition of (N, v), hard to answer in general
- $\Longrightarrow$ subsets of games (*classes of games*)

Classes of games

"Bigger coalition is better."

Definition

A cooperative game (N, v) is

- *monotonic* if for every $T \subseteq S \subseteq N$ it holds
$$v(T) \leq v(S),$$
- *superadditive* if for every $S, T \subseteq N$ such that $S \cap T = \emptyset$ it holds
$$v(S) + v(T) \leq v(S \cup T),$$
- *convex* if for every $S, T \subseteq N$ it holds
$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

Classes of games

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Definition

A cooperative game (N, v) is

- *monotonic*

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- *convex*

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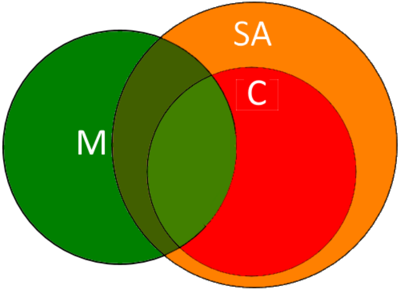


Yet another hierarchy

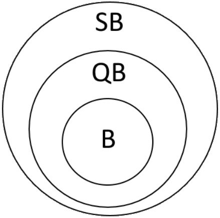
"Catch the core!"

Definition

A cooperative game (N, v) is

- *semibalanced* if $H(v) \neq \emptyset$
- *quasibalanced* if $CC(v) \neq \emptyset$
- *balanced* if $C(v) \neq \emptyset$

$C(v) \subseteq CC(v) \subseteq H(v)$



Bounds on claims

"Bounds on what I can claim."

1. b^v ... **utopia vector**
   - $b_i^v := v(N) - v(N \setminus i)$
   - *If I demand more, nobody cares...*
2. a^v ... **minimal right vector**
   - the real world is not an utopia: $\sum_{j \in N} b_j^v > v(N)$
   - take what you want, i take the rest...
   - $a_i^v := \max_{S, i \in S} v(S) - \sum_{j \in S \setminus i} b_j^v$

Bounds and cores and compromise

"In a view of the core.."

For $x \in C(v)$,

- $a_i^v \leq x_i \leq b_i^v$

For (N, v) a quasibalanced game,

- $a^v(N) \leq v(N) \leq b^v(N)$

Pick an efficient compromise...

Definition

the τ-**value** $\tau(v)$ of game (N, v) is defined as the unique convex combination of a^v and b^v such that $\sum_{i \in N} \tau(v)_i = v(N)$.

The values ϕ,n and τ

"To be fair, how fair are you?"

They **are** fair...:

- ϕ is frequently used as a fair solution concept (reasons already discussed)
- τ-value also chosen as a fair solution in several applications
- n is *fair* from point of view of one fairness predicate
    - it is a core selector ($C(v) \neq \emptyset \implies n(v) \in C(v)$)

...are they not?

- ϕ and τ are often **not** core selectors
- in many games: $\phi(v) \neq n(v) \neq \tau(v)$
- Which one to choose?

Egalitarianism

"If I can, I share with you..."

Definition

A tuple (i, j, α, x) is a **bilateral transfer** if

$$x_i - \alpha \geq x_j + \alpha.$$

- i, j ... me and you
- $x \in I(v)$ ... what we get
- $\alpha \geq 0$ ... what I share

Egalitarian core

"... but it must be a stable transfer."

Definition

An imputation $x \in C(v)$ is **egalitarian** if *no* $y \in C(v)$ is the result of any (i, j, α, x).

"No matter what you do, this is the best..."

Definition

An imputation $x \in C(v)$ is **strongly egalitarian** if *no* $y \in C(v)$ is the result of a finite sequence of bilateral transfers.

Differences in definitions

egalitarian $x \in C(v)$

- exists if $C(v) \neq \emptyset$
- more solutions
- $SE \subseteq E$

strongly egalitarian $x \in C(v)$

- unique solution
- solution of least squares:
- $\min_{y \in C(v)} \|y\|_2$

C_e as a fairness concept

"Fair and sane, however..."

1. **fair** thanks to *bilateral transfers*
2. **sane** thanks to *core* stability

Example

2-players game (N, v) where $v(1) = 1, v(2) = 0$ and $v(12) = 2$.

$C_e(v) = \{(1,1)^T\}$ ... why should 1 cooperate?

$\phi(v) = (1.5, 0.5)^T$ ... this is *more fair*

- One might say: "Its overdoing fairness..."

Inequity aversion

"How does it hurt, when I am better of?"

Definition

A players **inequity aversion utility** in the imputation x is

$$u_i(x) = x_i - \alpha_i \cdot \sum_{j \neq i} \max\{0, x_j - x_i\}.$$

- you feel like you lose α_i for 1 unit of j's advantage over you
- u_i remains to you, if count in the losses

"I can't stand to be the one *better of*!"

Definition

A players **inequity aversion utility** in the imputation x is

$$u_i(x) = x_i - \alpha_i \cdot \sum_{j \neq i} \max\{0, x_j - x_i\} - \beta_i \cdot \sum_{j \neq i} \max\{0, x_i - x_j\}.$$

Inequity aversion core

"In context of core stability..."

Definition

An **inequity aversion core** is a set of imputations $x \in C(v)$ such that for no $y \in C(v)$, there is a player i with

$$u_i(y) > u_i(x).$$

Example of inequity aversion

Example

2-player game (N, v) where $v(1) = a, v(2) = b$ and $v(1,2) = a + b + c,\ a \leq b$

- inequity $b - a$ before cooperation
- decision to cooperate $\implies$ distribute $(a + c_a, b + c_b)$
  - $c_a + c_b = c$
- inequity change $c_b - c_a = c - 2c_a$
- if $c_a < \alpha_1 \cdot (c - 2c_a)$
  - c_a ... what player 1 gets by cooperation
  - $\alpha_1 \cdot (c - 2c_a)$ ... what he feels he loses
  - if "$<$" happens $\implies$ won't cooperate
- $\alpha_1 = 0.25 \implies$ cooperation $\iff c_a \geq \frac{1}{6}c$
- $\alpha_1 = 1 \implies$ cooperation $\iff c_a \geq \frac{1}{3}c$
- $\alpha_1 =$ "∞" $\implies$ cooperation $\iff c_a \geq \frac{1}{2}c$

Disadvantage of IA

"All that matters is my aversion..."

- c_a was dependend on a, b.
- $a \ll b \implies$ same scenario as $a = b$

Presumption: "All players are *equal*."

Fairness predicates

"Division of solution concepts into elementary properties..."

Definition

A **predicate on the imputation space** of a cooperative *n*-person game is a mapping $\mathcal{P}$ that assigns every game (N, v) a subset $\mathcal{P}(v) \subseteq I(v)$.

Fairness Predicates

- subset of $I(v)$
- **does not** have to *make sense* on itself:
- Dummy player predicate *DP*
  - rules out $x \in I(v) : x_i > 0$ for i with contribution 0
  - not much of a concept

Solution concept

- subset of $I(v)$ (usually)
- **does** have to *make sense* on itself:
- Shapley value
  - *fair* distribution of payoff given by rules (EFF, ADD, DP, SYM)
  - an interesting concept

Fairness predicates

"Axioms as predicates..."

A (partial) one-point solution concept $\mathcal{P}$ satisfies

- **anonymity** if for any permutation σ of the player set N we have $\mathcal{P}(v)_i = \mathcal{P}(\sigma(v))_{\sigma(v)}$
- **additivity** if for two cooperative n-person games (N, v) and (N, w) the equation $\mathcal{P}(v + w) = \mathcal{P}(v) + \mathcal{P}(w)$ holds.
  - $\mathcal{P}(v) \neq \emptyset$ and $\mathcal{P}(w) \neq \emptyset$

A predicate $\mathcal{P}$ on the imputation space of cooperative n-person games

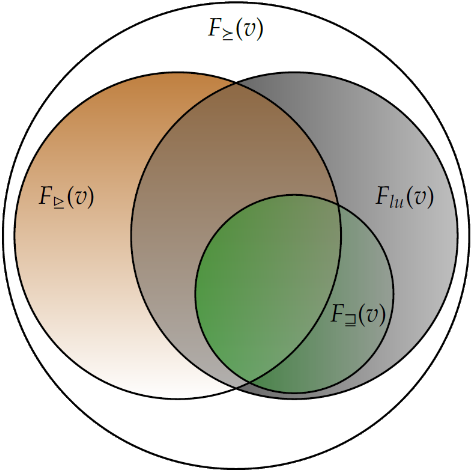
- **split** if for all (N, v) we have $\mathcal{P}(v_0) + s(v) = \mathcal{P}(v)$
  - $s(v)_i = v(i)$

"We are interested if solution concepts satisfy predicates..."

Fairness based on desirability

"If you work hard, you should get more."

4 desirability predicates:



Desirability of players $F_{\succeq}(v)$

"If you work hard, you should get more."

Definition

Player desirability relation $i \succeq j$ denotes that player i is more desirable than j, i.e.

$$v(A \cup \{i\}) \geq v(A \cup \{j\}) \text{ for } A \subseteq N \setminus i, j.$$

Definition

Player desirability-fair imputation $x \in I(v)$ is such that

$$i \succeq j \implies x_i \geq x_j.$$

The set of all such x is denoted by $F_{\succeq}(v)$.

$F_{\succeq}(v)$ and solution concepts

"If only I had time, I would convince you..."

Theorem

For a game (N, v), following hold.

1. $Ker(v) \subseteq F_{\succeq}(v)$
2. $n(v) \in F_{\succeq}(v)$
3. (N, v) *is quasi-balanced* $\implies$ τ*-value* $\tau(v) \in F_{\succeq}(v)$,
4. (N, v) *super-additive* $\implies$ *Shapley value* $\phi(v) \in F_{\succeq}(v)$,
5. *If* $C(v) \neq \emptyset \implies C(v) \cap F_{\succeq}(v) \neq \emptyset$,
6. *If* $C(v) \neq \emptyset \implies \emptyset \neq C_e(v) \subseteq F_{\succeq}(v)$.

Open questions:

- What about other solution concepts? (Bargaining set, Prekernel, ...)
- What are full characterisations of 3.,4.
- ...

$F_{\succeq}(v)$ and Core

"... at least something." If $C(v) \neq \emptyset \implies C(v) \cap F_{\succeq}(v) \neq \emptyset$

Idea:

$F_{\succeq}(v)$ and Core

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Idea:

- $x \in C(v)$

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Idea:

- $x \in C(v)$

1. if $i \succeq j$ and $x_i < x_j$
   - switch: $y_j = x_i$ and $y_i = x_j$

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2. if $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

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   - redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

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- $y \in C(v)$

$F_{\succeq}(v)$ and Core

1. If $i \succeq j$ and $x_i < x_j$, switch: $y_j = x_i$ and $y_i = x_j$

$y_i := x_j$
$y_j := x_i$
$y_k := x_k$ for $k \in N \setminus \{i, j\}$

Proof.

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Proof.

1. $x \in I(v) \implies y \in I(v)$

2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

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1. $x \in I(v) \implies y \in I(v)$
   1.1 (efficiency) $y(N) = v(N)$
   1.2 (individual rationality) $y_k \geq v(k)$ for $k \in N$
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      - $= y_1 + \cdots + y_{i-1} + y_i + y_{i+1} + \ldots, y_{j-1} + y_j + y_{j+1} + \cdots + y_n =$

   1.2 (individual rationality) $y_k \geq v(k)$ for $k \in N$

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   - $x_1 + \cdots + x_{i-1} + x_j + x_{i+1} + \cdots + x_{j-1} + x_i + x_{j+1} + \cdots + y_n =$
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      - $y_j \geq v(j) : y_j = x_i \geq v(i) \geq v(j)$

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1. $x \in I(v) \implies y \in I(v)$ (PROVED)
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

□

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   2.1 $i, j \in S$ and $i, j \notin S$

   2.2 $i \in S$ and $j \notin S$

   2.3 $i \notin S$ and $j \in S$

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      - $y(S) = x(S) \geq 0$

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      - $y(S) = x(S) \geq 0$

   2.2 $i \in S$ and $j \notin S$
      - $v(S) \leq x(S) = x_i + x(S \setminus i) < x_j + x(S \setminus i) = y(S)$

   2.3 $i \notin S$ and $j \in S$

□

$F_{\succeq}(v)$ and Core

1. If $i \succeq j$ and $x_i < x_j$, switch: $y_j = x_i$ and $y_i = x_j$

$y_i := x_j$
$y_j := x_i$
$y_k := x_k$ for $k \in N \setminus \{i, j\}$

Proof.

1. $x \in I(v) \implies y \in I(v)$ (PROVED)
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$
   - 2.1 $i, j \in S$ and $i, j \notin S$
     - $y(S) = x(S) \geq 0$
   - 2.2 $i \in S$ and $j \notin S$
     - $v(S) \leq x(S) = x_i + x(S \setminus i) < x_j + x(S \setminus i) = y(S)$
   - 2.3 $i \notin S$ and $j \in S$
     - $v(S) = v((S \setminus j) \cup j) \leq v((S \setminus j) \cup i) \leq x((S \setminus j) \cup i) = y(S)$

□

$F_{\succeq}(v)$ and Core

"... at least something." If $C(v) \neq \emptyset \implies C(v) \cap F_{\succeq}(v) \neq \emptyset$

Idea:

- $x \in C(v)$

1. if $i \succeq j$ and $x_i < x_j$ (PROVED)
   - switch: $y_j = x_i$ and $y_i = x_j$
2. if $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)
   - redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

- $y \in C(v)$

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

□

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

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Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$

2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

□

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$
   1.1 (efficiency) $y(N) = v(N)$

2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

□

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$
   1.1 (efficiency) $y(N) = v(N)$
   1.2 (individual rationality) $y_k \geq v(k)$ for $k \in N$
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

□

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$
   1.1 (efficiency) $y(N) = v(N)$
       - $y(N) = \sum_{i \in \Sigma} \frac{x(\Sigma)}{|\Sigma|} + \sum_{i \in N \setminus \Sigma} x_i =$

   1.2 (individual rationality) $y_k \geq v(k)$ for $k \in N$

2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

□

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

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Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$
   1.1 (efficiency) $y(N) = v(N)$
      - $y(N) = \sum_{i \in \Sigma} \frac{x(\Sigma)}{|\Sigma|} + \sum_{i \in N \setminus \Sigma} x_i =$
      - $= x(\Sigma) + x(N \setminus \Sigma) = x(N) = v(N)$

   1.2 (individual rationality) $y_k \geq v(k)$ for $k \in N$

2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

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$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

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Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i,j\}$

1. $x \in I(v) \implies y \in I(v)$
   1.1 (efficiency) $y(N) = v(N)$
      - $y(N) = \sum_{i \in \Sigma} \frac{x(\Sigma)}{|\Sigma|} + \sum_{i \in N \setminus \Sigma} x_i =$
      - $= x(\Sigma) + x(N \setminus \Sigma) = x(N) = v(N)$

   1.2 (individual rationality) $y_k \geq v(k)$ for $k \in N$
      - $i \in \Sigma : v(i) = \min_{j \in \Sigma} v(j) \leq \min_{j \in \Sigma} x_j \leq \frac{x(\Sigma)}{|\Sigma|} = y_i$

2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

□

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$
   1.1 (efficiency) $y(N) = v(N)$
      - $y(N) = \sum_{i \in \Sigma} \frac{x(\Sigma)}{|\Sigma|} + \sum_{i \in N \setminus \Sigma} x_i =$
      - $= x(\Sigma) + x(N \setminus \Sigma) = x(N) = v(N)$
   1.2 (individual rationality) $y_k \geq v(k)$ for $k \in N$
      - $i \in \Sigma : v(i) = \min_{j \in \Sigma} v(j) \leq \min_{j \in \Sigma} x_j \leq \frac{x(\Sigma)}{|\Sigma|} = y_i$
      - $i \notin \Sigma : v(i) \leq x_i = y_i$
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

□

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$ (PROVED)
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$

$F_{\succeq}(v)$ and Core

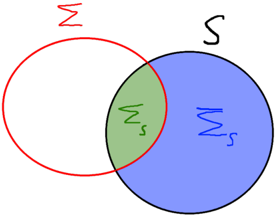
2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

Idea: $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$

1. $x \in I(v) \implies y \in I(v)$ (PROVED)
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$
   - $S = \Sigma_S + \overline{\Sigma}_S$



   -

$F_{\succeq}(v)$ and Core

2. $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$)

redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

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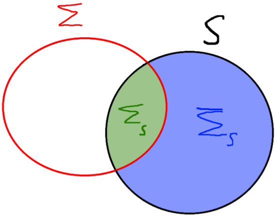
1. $x \in I(v) \implies y \in I(v)$ (PROVED)
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$
   - $S = \Sigma_S + \overline{\Sigma}_S$
   - $\Sigma_S = S \cap \Sigma$
   - 

$F_{\succeq}(v)$ and Core

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redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

Proof.

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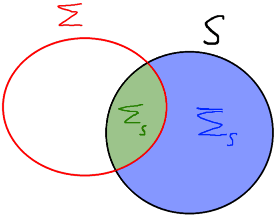
1. $x \in I(v) \implies y \in I(v)$ (PROVED)
2. $x(S) \geq v(S) \implies y(S) \geq v(S)$
   - $S = \Sigma_S + \overline{\Sigma}_S$
   - $\Sigma_S = S \cap \Sigma$
   - $\overline{\Sigma}_S = S - \Sigma_S$



   -

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

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2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$
    - no change outside Σ

$F_{\succeq}(v)$ and Core

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$F_{\succeq}(v)$ and Core

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1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$
   - no change outside Σ
3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - Σ_i for $i \in N$
     - i *smallest* players from Σ ordered by x:
     - $\Sigma = \{\sigma_1, \ldots, \sigma_k\} = \{e_1, \ldots, e_k\}$
     - $i \leq j \implies x(e_i) \leq x(e_j)$
     - $\Sigma_i = \{e_1, \ldots, e_i\}$
   - $a = |\Sigma_S|$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$
   - no change outside Σ
3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$
   - no change outside Σ
3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$
4. $y(\Sigma_a) + x(\overline{\Sigma}_S) \geq x(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_a) \geq x(\Sigma_a)$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

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   - no change outside Σ
3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$
4. $y(\Sigma_a) + x(\overline{\Sigma}_S) \geq x(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_a) \geq x(\Sigma_a)$
     - a times average of Σ is larger than first a elements of Σ
     - with respect to x

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$
   - no change outside Σ
3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$
4. $y(\Sigma_a) + x(\overline{\Sigma}_S) \geq x(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_a) \geq x(\Sigma_a)$
     - a times average of Σ is larger than first a elements of Σ
     - with respect to x

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
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   - no change outside Σ
3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$
4. $y(\Sigma_a) + x(\overline{\Sigma}_S) \geq x(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_a) \geq x(\Sigma_a)$
     - a times average of Σ is larger than first a elements of Σ
     - with respect to x
5. $x(\Sigma_a) + x(\overline{\Sigma}_S) \geq v(\Sigma_a \cup \overline{\Sigma}_S)$ $(x \in C(v))$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$
   - no change outside Σ
3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$
4. $y(\Sigma_a) + x(\overline{\Sigma}_S) \geq x(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_a) \geq x(\Sigma_a)$
     - a times average of Σ is larger than first a elements of Σ
     - with respect to x
5. $x(\Sigma_a) + x(\overline{\Sigma}_S) \geq v(\Sigma_a \cup \overline{\Sigma}_S)$ $(x \in C(v))$
6. $v(\Sigma_a \cup \overline{\Sigma}_S) = v(\Sigma_S \cup \overline{\Sigma}_S) = v(S)$

$F_{\succeq}(v)$ and Core

$x(S) \geq v(S) \implies y(S) \geq v(S)$

1. $y(S) = y(\Sigma_S) + y(\overline{\Sigma}_S)$
2. $y(\Sigma_S) + y(\overline{\Sigma}_S) = y(\Sigma_S) + x(\overline{\Sigma}_S)$
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3. $y(\Sigma_S) + x(\overline{\Sigma}_S) = y(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_S) = y(\Sigma_a)$
     - $\forall i, j \in \Sigma : y_i = y_j$
4. $y(\Sigma_a) + x(\overline{\Sigma}_S) \geq x(\Sigma_a) + x(\overline{\Sigma}_S)$
   - $y(\Sigma_a) \geq x(\Sigma_a)$
     - a times average of Σ is larger than first a elements of Σ
     - with respect to x
5. $x(\Sigma_a) + x(\overline{\Sigma}_S) \geq v(\Sigma_a \cup \overline{\Sigma}_S)$ $(x \in C(v))$
6. $v(\Sigma_a \cup \overline{\Sigma}_S) = v(\Sigma_S \cup \overline{\Sigma}_S) = v(S)$
   - Σ are **substitutes**:
   - $i \succeq j$ and $j \succeq i \implies v(S \cup i) = v(S \cup j)$ for $S \setminus \{i, j\}$ $\square$

$F_{\succeq}(v)$ and Core

"... at least something." If $C(v) \neq \emptyset \implies C(v) \cap F_{\succeq}(v) \neq \emptyset$

Idea:

- $x \in C(v)$

1. if $i \succeq j$ and $x_i < x_j$ (PROVED)
   - switch: $y_j = x_i$ and $y_i = x_j$
2. if $\Sigma = \{i_1, \ldots, i_k\}$ substitutes (i.e. $i \succeq j$ and $j \succeq i$) (PROVED)
   - redistribute: $i \in \Sigma \implies y_i = \frac{x(\Sigma)}{|\Sigma|}$

- $y \in C(v)$

$F_{\succeq}(v)$ and solution concepts

"If only I had time, I would convince you..."

Theorem

For a game (N, v), following hold.

1. $Ker(v) \subseteq F_{\succeq}(v)$
2. $n(v) \in F_{\succeq}(v)$
3. (N, v) *is quasi-balanced* $\implies$ τ*-value* $\tau(v) \in F_{\succeq}(v)$,
4. (N, v) *super-additive* $\implies$ *Shapley value* $\phi(v) \in F_{\succeq}(v)$,
5. *If* $C(v) \neq \emptyset \implies C(v) \cap F_{\succeq}(v) \neq \emptyset$,
6. *If* $C(v) \neq \emptyset \implies \emptyset \neq C_e(v) \subseteq F_{\succeq}(v)$.

Weak Desirability of players $F_{\trianglerighteq}(v)$

"I don't know if it holds, but I feel like it does..."

desirability: $i \succeq j \implies v(A \cup \{i\}) \geq v(A \cup \{j\})$ for $A \subseteq N \setminus i, j$
# of conditions: $2^{|N|-2}$

Problem:

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Problem:

- infeasible to check for relatively small number of players

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Problem:

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- Solution: pick a subset of conditions

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$\#$ of conditions: $2^{|N|-2}$

Problem:

- infeasible to check for relatively small number of players
- Solution: pick a subset of conditions
  - *individual payoffs* and *marginal contributions* to N

Weak Desirability of players $F_{\trianglerighteq}(v)$

1. *individual payoffs*

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Definition

Player weak desirability relation $i \trianglerighteq j$ denotes that player i is more desirable (in a weak sense) than j, i.e.

$$v(i) \geq v(j) \text{ and } v(N \setminus i) \leq v(N \setminus j).$$

Definition

Weak player desirability-fair imputation $x \in I(v)$ is such that

$$i \trianglerighteq j \implies x_i \geq x_j.$$

The set of such x is denoted by $F_{\trianglerighteq}(v)$.

$F_{\trianglerighteq}(v) \subseteq F_{\succeq}(v)$

"It takes less to get me started..."

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- therefore, it is *activated* more often
- $\succeq$ holds for at least as much pairs of players as $\trianglerighteq$

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- Example:
  - $i_1 \trianglerighteq i_2,\ i_3 \trianglerighteq i_4 \implies x_{i_1} \geq x_{i_2}, x_{i_3} \geq x_{i_4}$
  - $i_3 \succeq i_4 \implies x_{i_3} \geq x_{i_4}$

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- Consequence: $F_{\trianglerighteq}(v) \subseteq F_{\succeq}(v)$

$F_{\trianglerighteq}(v)$ and solution concepts

"Is it interesting? Nobody knows yet..."

Theorem

For a game (N, v), following hold:

1. *(N, v) is 1-convex $\implies \tau(v) \in F_{\trianglerighteq}(v) \cap C(v)$,*
2. *(N, v) is quasi-balanced and a little condition $\implies \tau(v) \in F_{\trianglerighteq}(v)$.*

Open questions:

- **basically the rest!**

Desirability relation on coalitions $F_{\sqsupseteq}(v)$

"United we stand, divided we fall..."

Definition

Desirability relation on coalitions $A \sqsupseteq B$ denotes coalition A is more desirable than B, i.e.

$$v(C \cup A) \geq v(C \cup B) \text{ for all } C \subseteq N \setminus (A \cup B).$$

Definition

Coalition desirability-fair imputation $x \in I(v)$ is such that

$$A \sqsupseteq B \implies x(A) \geq x(B).$$

The set of such x is denoted by $F_{\sqsupseteq}(v)$.

Desirability relation on coalitions $F_{\sqsupseteq}(v)$

"But we actually mostly fall..."

- $i \succeq j \iff \{i\} \sqsupseteq \{j\}$
- $F_{\sqsupseteq}(v) \subseteq F_{\succeq}(v)$
- exists game (N, v):
  - $F_{\sqsupseteq}(v) \cap C(v) = \emptyset$
  - $\tau(v) \notin F_{\sqsupseteq}(v)$
  - $\phi(v) \notin F_{\sqsupseteq}(v)$
  - $n(v) \notin F_{\sqsupseteq}(v)$
- Banktruptcy games: Aristotelian proportional division
  - $x = \frac{E}{d_1 + \cdots + d_n}(d_1, \ldots, d_n)$
  - $x \in F_{\sqsupseteq}(v)$

Desirability of equivalence classes $F_{lu}(v)$

"Getting $\sqsupseteq$ weaker by labor unions..."

- same problem as for $\succeq$:
  - 2^N coalitions
  - many of them *unlikely*
- Task: select a sensible subset of condition
  - coalition of substitutes K (*labor union*)
  - $K \sqsupseteq \{i\}$ (*factory owner* i)
  - $x(K) \geq x_i$ (K: "We are not slaves!")

Definition

The labor union-fair imputation $x \in I(v)$ is such that

1. $K \sqsupseteq \{i\} \implies x(K) \geq x_i$,
2. $x \in F_{\succeq}(v)$.

The set of such x is denoted by $F_{lu}(v)$.

Desirability of equivalence classes $F_{lu}(v)$

"At least the egalitarian core C_e is fair for the workers."

Theorem

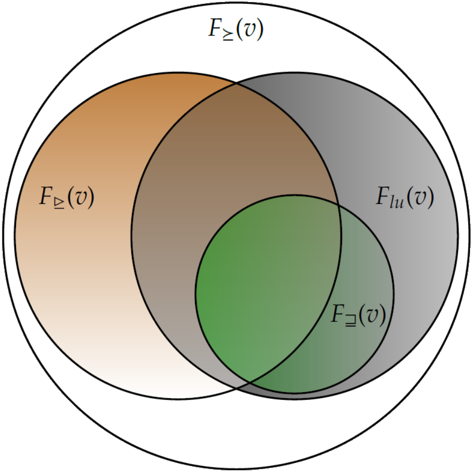
$C_e \subseteq F_{lu}(v)$ for convex games (N, v).

Also, minor results about Shapley, τ-value and nucleolus.

Fairness based on desirability

"If you work hard, you should get more."

4 desirability predicates:



Core-satisfiability

"This is fair, and that is fair, so which one is more fair?"

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   - Do these games really matter?

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Definition

A predicate $\mathcal{P}$ is **satisfiable within the core** (in a class G) if

$$(N, v) \in G : C(v) \neq \emptyset \implies \mathcal{P}(v) \cap C(v) \neq \emptyset.$$

We say $\mathcal{P}$ is *core-satisfiable* or simply *satisfiable*.

Core-satisfiability

"It is good, at least when the game is stable."

Definition

A predicate $\mathcal{P}$ is **satisfiable within the core** (in a class G) if

$$(N, v) \in G : C(v) \neq \emptyset \implies P(v) \cap C(v) \neq \emptyset.$$

- we can define different ?-satifiability
- Core-satisfiability enoforces stability of the solution

Core-satisfiability

"And how does it look, from the core point-of-view?"

Theorem

1. $F_{\succeq}(v)$ *is satisfiable for every game,*
2. $F^0_{\succeq}(v)$ *is satisfiable for every game,*
3. $F_{\trianglerighteq}$ *is satisfiable for every convex and* 1*-convex game,*
4. $F_{\trianglerighteq}$ *is* **not** *satisfiable for every superadditive game,*
5. F_{lu} *is satisfiable for every convex game, but* **not** *every superadditive game.*

Individual or Culture Specific Notions of Fairness

"This is fair to you?"

- the most natural setting
  - not only different interests
  - but also notions of fairness
- modification in the *stability* notion (different from Core)

Modified stability condition

"The core sounds fine, but lets keep it sensible..."

imputation $x \in C(v)$ if

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- my fairness notion = my **culture** (cultural identification)
- How does our cultural differences affect us?

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   2.2 $x(A) \geq v_S(A)$ for every $A \subseteq S$ **culturally compatible**

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Definition

Let (N, v) be a cooperative game and let $CC(v)$ be the set of its culturally compatible coalitions.

A **culturally compatible core** C_{cc} is

$$C_{cc}(v) = \{x \in \cap_{i \in N} F_i(v) \big| x(N) = v(N) \text{ and } x(A) \geq v(A), \forall A \in CC(v)\}.$$

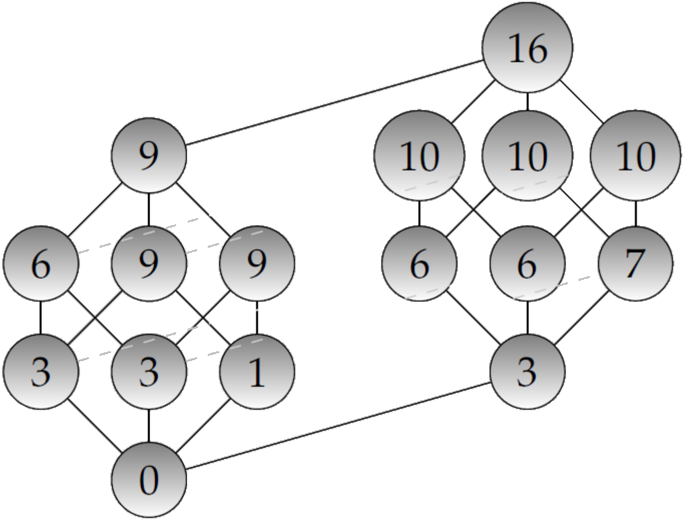
Example of the model - general

"I won't believe it until I see it..."

- $N = \{1, 2, 3, 4\}$
- $F_1 = F_2 = F^0_{\succeq}(v), F_3 = C_e, F_4 = \phi(v)$
  - $P(v_0) + s(v) = P(v)$ **split**
- Players 1 and 2 are mutually culturally compatible in
  - every zero-normalised 2-player subgame $v_{\{1,2\}}$
- Players $1, 2, 4$ are culturally compatible in 3-player subgame which is
  - zero-normalised: $v_0 = v$, $s(v) = 0$
  - $\phi(v) \in C(v)$
- Players $1, 2, 3$ are culturally compatible in 3-player subgame which is
  - zero-normalised
- Player $3, 4$ are mostly uncompatible $(\phi(v) \notin C_e(v))$

Example of the model - general

"Give me a real example!"



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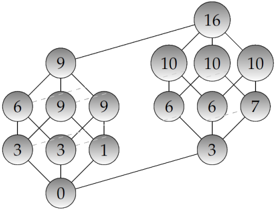
- 

- Subgame $(N, v_{\{1,2,3\}})$ has an empty core
- blocking coalitions $\{1,3\}$ and $\{2,3\}$
- They are **not** culturally compatible
  - $v(13) = v(23) = 9$
  - for player 3: $F_3 = \{(4.5, 4.5)\}$
  - for players 1, 2: $F_1 = F_2 = \{(5.5, 3.5)\}$
- therefore $(3,3,3) \in C_{cc}(v_{\{1,2,3\}})$
- paradoxically: cultural incompatibility $\implies$ cultural compatibility

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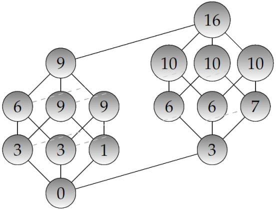
- 

- $\phi(v) = (4, 4, 4, 4)$
- $\phi(v) \in C_e$
- $\phi(v) \in F_{\succeq}(v)$
- $\implies (4, 4, 4, 4) \in C_{cc}(v)$