Egalitarian solutions in the core

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Summary

- Recap of TU games once again
- Reduced game properties
- Davis-Maschler reduced game property
- Converse reduced game property
- Constrained egalitarian property
- Egalitarian orderings in core
- Lorenz stable set
- Leximin stable allocation
- Surplus
- Egalitarian core

Recap

- the set of players $N = \{1...n\}$
- characteristic function $v : 2^N \to \mathbb{R}$
  - $v(\emptyset) = 0$
- Cooperative game (N, v)
- Payoff vector x, where x_i is profit of player i
- Game classes=sets of games
  - Γ set of all games
  - Γ_0 set of games on which the solution is defined
  - Γ_c games with nonempty core
- solution is a function $\sigma\colon \Gamma_0 \to 2^{\mathbb{R}^n}$

Recap of restrictions

- core of a game (N,v) is defined as

$$C(v) = \{x \in \mathbb{R}^n | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N\}$$

- imputations (or allocations) are payoff vectors that are efficient and individualy rational

Reduced game

Definition

David-Maschler reduced game is for a game $(N, v) \in \Gamma_0$ and non-empty coalition T and vector $x \in \mathbb{R}^n$ defined as: (T, v_x^T)

$$v_x^T(S) = \begin{cases} 0 & S = \emptyset \\ v(N) - x(N \setminus S) & S = T \\ \max\limits_{Q \subset N \setminus T} \{v(S \cup Q) - x(Q)\} & \forall S \subset T, S \neq \emptyset \end{cases}$$

Example

Have a game with $N = \{1,2,3\}, v(1,2) = 1, v(1,3) = 1, v(2,3) = 2, v(1,2,3) = 2, \textit{other} = 0$ and vector $x = (1/2, 1/2, 0)$

Then for $T = \{1,2\}$

- $v_x^T(\{1\}) = 1, (v(1,3) - x(3))$
- $v_x^T(\{2\}) = 2, (v(2,3) - x(3))$
- $v_x^T(\{1,2\}) = 2, (v(1,2,3) - x(3))$
- we suppose that player 3 is happy with vector x

Davis-Maschler reduced game property

A solution σ satisfies DM-RGP if for $(N, v) \in \Gamma_0, x \in \sigma(N, v)$ holds.

$(S, v_x^S) \in \Gamma_0$, $x^S \in \sigma(S, v_x^S)$

For the example reduced games with player 3 would make it not DM-RGP

Converse reduced game property

A solution σ satisfies CRGP if for a set $L = \{S \subseteq N : |S| = 2\}$ holds:

$\forall S \in L, (S, v_x^S) \in \Gamma_0 \wedge x^s \in (S, v_x^S) \implies x \in \sigma(N, v)$

Similarity with CG-completeness

Constrained egalitarian property

A solution σ defined on Γ_c satisfies CEP, if every 2-person game in Γ_c satisfies CEP.

Can happen only if 2-person games are superadditive

2-person solution

$(\{i, j\}, v)$ lets say that $v(i) \leq v(j)$ then the solution is.

$$CE_j(\{i, j\}, v) = \max\{\frac{v(ij)}{2}, v(j)\}$$

$$CE_i(\{i, j\}, v) = v(ij) - CE_j(\{i, j\}, v)$$

Egalitarian orderings in core

- Lorenz binary relation $\succ_L$
- Leximin binary relation $\succ_m$

for $x \in \mathbb{R}^N, \hat{x}$ is obtained by rearranging coordinates of x in a non-decreasing order so. $\hat{x}_1 \leq \cdots \leq \hat{x}_n$

Lorenz binary relation $\succ_L$

for $x, y \in \mathbb{R}^n$ we say that $x \succ_L y$ when $\forall k$:

$$\sum_{j=1}^{k} \hat{x}_j \geq \sum_{j=1}^{k} \hat{y}_j$$

$\exists k$ holds

$$\sum_{j=1}^{k} \hat{x}_j > \sum_{j=1}^{k} \hat{y}_j$$

Examples

Suppose the vectors are in the core

$(3,3,3,0),(4,2,2,1)$, are not comparable

$(3,3,3,0),(4,2,2,1) \succ_L (5,2,2,0)$

Lorenz stable set

we denote

$$LSS(N, v) = \{x \in C(N, v) : \nexists y \in C(N, v), y \succ_L x\}$$

The less wealth good players get the better.

Leximin binary relation $\succ_m$

for $x, y \in \mathbb{R}^n$ we say that x leximin dominates y when $\exists k \in \{1 \ldots n-1\}$ such

$$\forall i \in \{1 \ldots k\}, \hat{x}_i = \hat{y}_i \land \hat{x}_{k+1} > \hat{y}_{k+1}$$

Examples

Suppose the vectors are in the core

$(4, 2, 2, 1) \succ_m (3, 3, 3, 0)$

Vectors are not comparable if and only if they are equal. (Linear ordering)

The more wealth worse players get the better.

Leximin stable allocation

We denote

$$\{LSA(N,v)\} = \{x \in C(N,v) : \forall y \in C(N,v), y \neq x, x \succ_m y\}$$

Theorem 1.1

Lorenz stable set is non-empty and in general not single valued for stable games.

LSS contains LSA (let's say y)
Because for every other $x \in C(N, v)$ there exists index k, before which $x_j = y_j$ and $y_j > x_j \implies \nexists b \in C(N, v), b \succ_L y$

Theorem 1.2

LSA is in general not single valued for stable games. Example

Let (N,v) be a 4-person balanced game

$$v(S) = \begin{cases} 6 & S \in \{\{1,2\},\{1,3\}\} \\ 8 & S = \{1,2,3\} \\ 9 & S = N \\ 0 & other \end{cases}$$

$LSA = (4,2,2,1)$ and so it is in LSS, also $(3,3,3,0) \in LSS$
else the core restrictions would be violated
we can see that the two vectors are $\succ_L$ incomparable

Theorem 2.1

Lemma

$\forall (N, v) \in \Gamma_c, \forall T \subset N, \forall y \in C(N, v), x \in C(T, v_y^T) \implies (x, y^{N \setminus T}) \in C(N, v)$

Not in scope of this presentation - papers

Core satisfies DM-RGP

Theorem 2

LSS satisfies CEP and DM-RGP (similarly can be proven that LSA satisfies CEP and DM-RGP)

Core satisfies DM-RGP

Let $x \in C(N, v) \implies x(T) + x(Q) >= v(T \cup Q)$ for $T \cap Q = \emptyset$

So we have that $x(T) >= v(T \cup Q) - x(Q)$

Let's have a reduced game (S, v_x^S) and let's say that $T \subseteq S$ and Q be the subset of $N \setminus S$ which makes the maximum coalition with T

Then by definition of reduced games

$x(T) >= v_x^S(T)$

LSS satisfies CEP

So lets have 2-person game with players i,j and let's say that $v(i) < v(j)$

LSS is in the core by definition, this means that the solution is efficient.

So $x_i + x_j = v(ij)$ then $CE_i = v(ij) - CE_j$ is trivial.

We know that $x_j >= v(j)$ from definition of core.

If $CE_j < v(ij)/2$ then $x_j < x_i \implies (v(ij)/2, v(ij)/2) \succ_L$ which means $x_j >= \max\{\frac{v(ij)}{2}, v(j)\}$

Now let's suppose that $x_j > \max\{\frac{v(ij)}{2}, v(j)\}$

Then again, $(CE_j, x_i + x_j - CE_j) \succ_L x$ which is a contradiction. That means $x_j = CEP_j$

LSS satisfies DM-RGP

Assuming that LSS does not satisfy DM-RGP.

$\implies \exists S \subset N \wedge x \in LSS(N, v), x^S \notin (S, v_S^y)$

Due to the fact that core satisfies DM-RGP $\exists z \succ_L x$

Have $y \in \mathbb{R}, j \in N \setminus S y_j = x_j, i \in S, y_i = z_i$

By the lemma we know $v(N) = v(y) \wedge y \in C(N, v)$

We see that $y \succ_L x \implies y \notin C(N, v)$

Contradiction.

Lemma

$\forall (N, v) \in \Gamma_c, \forall T \subset N, \forall y \in C(N, v), x \in C(T, v_y^T) \implies (x, y^{N \setminus T}) \in C(N, v)$

Surplus

For a game (N,v) and a payoff vector $x \in \mathbb{R}^n$ surplus of player i against player j is defined as:

$$s_{ij}(x, N.v) = \max_{i \in S, j \notin S, S \subset N} (v(S) - x(S))$$

It can be interpreted as how much can player i hope to gain without player j.

Egalitarian core

For a game (N,v), EC(N,v) is the set:

$$EC(N, v) = \{x \in C(N, v) | x_i > x_j \implies s_{ij}(x, N, v) = 0\}$$

By knowing the result (and by example) we get that $LSA \in LSS \subseteq EC$

Associations with last lecture

"Last time" we had that EC is a solution where the result of any bilateral transfer in $x \in C(N, v)$ does not belong in Core.

If we look at the definition of surplus

$$s_{ij}(x, N.v) = \max_{i \in S, j \notin S, S \subset N} (v(S) - x(S))$$

and on core constraints

$\sum_{i \in S} x_i \geq v(S), \forall S \subseteq N$

we see that surpluses can be either negative or equal 0

Associations 2

Now if we consider a bilateral transfer between player i and j, where $x_i > x_j$

For coalitions with both j,i their worth does not change so they do not limit maximal transfer.

Then The most x_i can can give is the $-s_{ij}$, for coalitions S with player i but without j, i can give at most min(x(S)-v(S)) (similar to surplus value) otherwise Core constraints are violated.

We can see that $s_{ij} = 0$ implies that any bilateral transfer results outside of the Core

Results of the paper

Alternative definition of EC

Egalitarian core is a solution for balanced games that satisfies both CEP and DM-RGP

This is by reference equivalent to solution that satisfies CEP, DM-RGP and CRGP

Important relation

This can be seen, because, if $x \in LSS \wedge x \notin EC \implies$ there is a bilateral transfer in the core, the transfer would create a vector $y \in C(N, v) \wedge y \succ_L x$

$$LSA \in LSS \subseteq EC$$

Example

Let this be a 4-person balanced game.

$$v(S) = \begin{cases} 2 & S \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\} \\ 4 & S = N \\ 0 & other \end{cases}$$

(1,1,1,1) and (2,2,0,0) are in EC(N,v) but $(1,1,1,1) \succ_L (2,2,0,0)$

and due to core constraints $LSS(N,v) = \{(1,1,1,1)\}$