Cooperative game theory and bankruptcy problems

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Introduction to cooperative game theory

A few (safe) assumptions...

- Transferable utility (TU): utility can be **freely** distributed among players.
- Non-transferable utility (NTU).

In this lecture, we shall deal only with TU games.

Definition

A **cooperative (TU) game** is an ordered pair (N, v), where N is a finite set of players $\{1, 2, \ldots, n\}$ and $v\colon 2^N \to \mathbb{R}$ is the characteristic function of the cooperative game. We further assume that $v(\emptyset) = 0$.

We denote the set of n-person cooperative games by Γ^n. Subsets of N are called **coalitions** and N itself is called the **grand coalition**. We often write v instead of (N, v) whenever there is no confusion over what the player set is. We also shorten $v(\{i\}$ into $v(i)$.

Payoff vectors and solutions (solution concepts)

For a game (N, v), the vectors $x \in \mathbb{R}^n$ are **payoff vectors**, where x_i corresponds to the payoff given to the player i.

We are interested in **feasible** payoff vectors, that is $x \in \mathbb{R}^n$ such that $x(N) \leq v(N)$. (We often write $x(S)$ instead of $\sum_{i \in S} x_i$.)

Definition

Let Γ be a set of games. We say that a function σ is **solution** or **solution concept** if it associates with every game $(N, v) \in \Gamma$ a subset $\sigma(N, v)$ of feasible payoff vectors of (N, v).

- Collective rationality: $\sum_{i \in N} x_i = v(N)$
- Individual rationality: $x_i \geq v(\{i\})$ for every $i \in N$

Definition

An **imputation** of $(N, v) \in \Gamma^N$ is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for every $i \in N$. The set of all imputations of a given game (N, v) is denoted by $I(v)$.

In other words, imputations are collectively and individually rational payoff vectors.

Definition

The **core** of $(N, v) \in \Gamma^N$ is the set

$$C(v) = \Big\{ x \in I(v);\ \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N \Big\}.$$

No coalition has an incentive to leave the grand coalition, since it gets at least $v(S)$ in any vector of the core.

Downsides? The core can be empty. On the other hand, core can be "big" and players can prefer one outcome of the core to the other...

Definition

The Shapley value is the function $\phi\colon \Gamma^n \to \mathbb{R}^n$ such that

$$\phi_i(v) = \sum_{S \subseteq N, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}(v(S) - v(S \setminus i)), \forall i \in N.$$

Consider all possible orderings of n players in which the grand coalition can be formed and consider the marginal contribution of i in the moment of entering the formation. Take the average of such marginal contributions.

While the formula is interesting, there is also an axiomatic definition.

The Shapley value continued

There is a large program towards finding meaningful and desired properties by which we could axiomatise given solution concept. The Shapley value is one of the classical solution concepts from this point of view. We have:

- Uniqueness,
- fairness (in **some** sense),
- effectiveness.

Theorem (Shapley, 1952)

There exists a unique function $f : \Gamma^N \to \mathbb{R}^N$, satisfying the following properties for every $(N, v) \in \Gamma^N$.

- *(Efficiency) It holds that $\sum_{i \in N} f_i(v) = v(N)$.*
- *(Dummy player) It holds $f_i(v) = 0$ for every $i \in N$, such that for every $S \setminus \{i\} \subseteq N$, equality $v(S \cup \{i\}) = v(S)$ holds.*
- *(Symmetry) We have $f_i(v) = f_j(v)$ if for every $S \subseteq N \setminus \{i, j\}$,*

$$v(S \cup i) - v(S) = v(S \cup j) - v(S).$$

- *(Additivity) For every two games $u, v \in \Gamma^N$, $f_i(u + v) = f_i(u) + f_i(v)$ holds.*

Can you come up with a non-desired property of the Shapley value?

The *excess* is the value $e(S, x) := v(S) - x(S)$ which compares the worth of coalition S and the distribution of its players' payoff in imputation x and shows its remaining potential. Further, $\theta(x) \in \mathbb{R}^{2^{|N|}}$ is a vector of excesses with respect to x which is arranged in non-increasing order.

Definition

The **nucleolus** $n \colon \Gamma^n \to \mathbb{R}^n$ is the minimal imputation x with respect to the lexicographical ordering of $\theta(x)$ i.e.

$$\theta(x) < \theta(y) \text{ if } \exists k : \forall i < k : \theta_i(x) = \theta_i(y) \text{ and } \theta_k(x) < \theta_k(y).$$

The excess of S on x can be regarded as a "complaint" of S towards the imputation x.

With nucleolus we want to "minimize the biggest complaints".

Definition (introduced by Davis and Maschler, 1965)

For a game (N, v), the **maximum surplus** $s_{k,l}(x)$ of player k over agent l with respect to a payoff vector x is the maximum excess from a coalition that includes k bud does exclude l: $s_{k,l}(x) := \max_{S \subseteq N \setminus l, k \in S} e(S, x)$.

The maximum surplus measures "bargaining power" of one player over the other since it corresponds to the maximal value that k can get without the cooperation of l by withdrawing from the proposed payoff x under the assumption that all other players in S are happy with x.

Definition

The **kernel** of a game (N, v) is the set of imputations of $K(v)$ such that for any $x \in K(v)$ and any two $k, l \in N$, $k \neq l$, we have either

$$s_{k,l}(x) \geq s_{l,k}(x) \text{ or } x_k = v(k).$$

Each player can compare his bargaining power over any other player. Either his position for negotiation is not worse or he is immune to any threats since he cannot loose anything.

Definition

For a game (N, v), the **prekernel** is the set of imputations where any two players have the same bargaining power over each other.

Clearly, prekernel is a subset of kernel for each cooperative game.

2-person games

We have already seen such games in the first lecture when speaking about **contested garment problem**.

Definition

The **2-person game** is a cooperative game with player set of size 2.

Definition

The **standard solution** of a 2-person game v with player set $\{1, 2\}$ is given by

$$x_i = \frac{v(12) - v(1) - v(2)}{2} + v(i).$$

This coincides with the Shapley value, kernel, pre-kernel, and nucleolus on 2-person games!

More so, it is the only symmetric and efficient point-valued solution concept for 2-person games that is covariant under strategic equivalence. *(Explain if there is time.)*

Let us transpose this principle to the 2-creditor bankruptcy problem with estate E and claims d_1, d_2. The amount that each claimant i concedes to the other claimant j is $(E - d_i)_+$, where

$$\theta_+ = \max(\theta, 0).$$

The amount at issue is therefore

$$E - (E - d_1)_+ - (E - d_2)_+;$$

it is shared equally between the two claimants, and, in addition, each claimant receives the amount conceded to her by the other one. Thus the total amount awarded to i is

$$x_i = \frac{E - (E - d_1)_+ - (E - d_2)_+}{2} + (E - d_j)_+. \tag{2.1}$$

We will say that this division (of E for claims d_1, d_2) is prescribed by the CG (contested garment) principle.[7,8]

Back to bankruptcies

Definition

The **bankruptcy game** $v_{E,d}$ corresponding to the bankruptcy problem (E, d) is defined as $v_{E,d}(S) := (E - d(N \setminus S))_+$.

Remember again that for a real number Θ, we define $\Theta_+ := \max(\Theta, 0)$.

Worth of a coalition S is what it can get without going to court, in other words accepting the better outcome of getting either nothing or what is left of E after all players from $N \setminus S$ take their full claims.

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Definition

The **reduced game** $v^{S,x}$ is defined on the player set S for given v and payoff vector x as:

$$v^{S,x}(T) = \begin{cases} x(T) & \text{if } T = S \text{ or } T = \emptyset, \\ \max\{v(Q \cup T) - x(Q) : Q \subseteq N \setminus S\} & \text{if } \emptyset \subsetneq T \subsetneq S. \end{cases}$$

- In reduced game, the players in S decide how to divide between them the amount from x. The coalition S gets $x(S)$, the empty set nothing.
- We assume that the players outside of S get what x proposes.
- A proper subcoalition T of S can call partners from Q, formed by players outside of S, and then get $v(Q \cup T)$. It has to pay $x(Q)$ to Q first.

Our main goal

Theorem (D, Aumann and Maschler, 1985)

The consistent solution of a bankruptcy problem is the nucleolus of the corresponding game.

The proof is adapted (with some changes here and there) from the following paper:

- Aumann, R. J., & Maschler, M. (1985). *Game theoretic analysis of a bankruptcy problem from the Talmud*. Journal of Economic Theory, 36(2), 195-213.

We shall prove it in a sequence of (four) lemmas.

Lemma (L1, on restricted game and restricted bankruptcy problem)

Let x be a solution of the bankruptcy problem (E, d), such that $0 \le x_i \le d_i$ for all $i \in N$. Then for any coalition S,

$$v_{E,d}^{S,x} = v_{x(S),d\restriction S}.$$

The reduced bankruptcy game is the game corresponding to the "reduced bankruptcy problem".

Proof. Set $v := v_{E;d}$ and $v^S := v_{E;d}^{S;x}$. First let $\varnothing \subsetneqq T \subsetneqq S$, and let the maximum in the definition of $v^S(T)$ be attained at Q. Since $x_i \geqslant 0$ and $a_+ - b_+ \leqslant (a-b)_+$ for all a and b, we have

$$\begin{aligned} v^S(T) &= v(T \cup Q) - x(Q) = (E - d(N \backslash (Q \cup T)))_+ - (x(Q))_+ \\ &\leqslant (x(N) - d(N \backslash (Q \cup T)) - x(Q))_+ \\ &= [x(S) - d(S \backslash T) - (d-x)(N \backslash (S \cup Q))]_+ \\ &\leqslant (x(S) - d(S \backslash T))_+, \end{aligned} \tag{6.3}$$

where the last inequality follows from $x_i \leqslant d_i$. On the other hand, setting $Q = N \backslash S$ yields

$$\begin{aligned} v^S(T) &\geqslant v(T \cup (N \backslash S)) - x(N \backslash S) \\ &= (E - d(N \backslash (T \cup (N \backslash S))))_+ - (x(N) - x(S)) \\ &\geqslant (E - d(S \backslash T)) - (E - x(S)) = x(S) - d(S \backslash T); \end{aligned} \tag{6.4}$$

Lemma (L1, on restricted game and restricted bankruptcy problem)

Let x be a solution of the bankruptcy problem (E, d), such that $0 \leq x_i \leq d_i$ for all $i \in N$. Then for any coalition S,

$$v_{E,d}^{S,x} = v_{x(S),d\restriction S}.$$

Proof continues....

and setting $Q = \varnothing$ yields

$$v^S(T) \geqslant v(T \cup \varnothing) - x(\varnothing) = v(T) = (E - d(N \backslash T))_+ \geqslant 0. \qquad (6.5)$$

Formulas (6.4) and (6.5) together yield

$$v^S(T) \geqslant (x(S) - d(S \backslash T))_+ ;$$

together with (6.3), this yields

$$v^S(T) = (x(S) - d(S \backslash T))_+ = v_{x(S);d|S}(T). \qquad (6.6)$$

When $T = \varnothing$ or $T = S$, formula (6.6) is immediate, so the proof of the lemma is complete.

Lemma (L2, on the pre-kernel of restricted game)

Let x be in the pre-kernel of a game v and let S be a coalition with exactly two players. Then $x \restriction S$ is the standard solution of $v^{S,x}$.

In other words, if we take a vector in prekernel and restrict it to any two-player coalition, it will be the standard solution of the restricted bankruptcy game (and thank to the previous lemma, also of the restricted bankruptcy problem).

Proof: Let $S = \{i, j\}$. By the definition of the maximal surplus of i over j, we can write:

$$\begin{aligned} s_{i,j}(x) &= \max_{Q \subseteq N \setminus S} (v(Q \cup i) - x(Q \cup i)) \\ &= \max_{Q \subseteq N \setminus S} (v(Q \cup i) - x(Q) - x_i = v^{S,x}(i) - x_i. \end{aligned} \tag{1}$$

The same can be done for $s_{i,j}(x)$. The definition of the prekernel gives us that $s_{i,j}(x) = s_{j,i}(x)$ and therefore,

$$v^{S,x}(i) - x_i = v^{S,x}(j) - x_j.$$

Finally we get $x_i - x_j = v^{S,x}(i) - v^{S,x}(j)$ and $x_i + x_j = v^{S,x}(i, j)$. After rearranging, we conclude that indeed $x \restriction S$ is the standard solution of $v^{S,x}$. $\square$

Lemma (L3, on CG solution and 2-person bankruptcy game)

The contested garment solution of 2-person bankruptcy problem is the standard solution of the corresponding game.

Proof: The previous lemma (L2) and the facts we know about the standard solution of 2-person cooperative game and CG problem. □

Lemma (L4, on the kernel of bankruptcy game)

The kernel of a bankruptcy game $v_{E,d}$ consists of a single point, namely the consistent solution of the problem (E, d).

Proof: For simplicity, $v := v_{E,d}$. Let x be a vector in the kernel of v.

- We claim that the game v is superadditive, that is for every two disjoint coalitions S, T, $v(S) + v(T) \leq v(S \cup T)$. We can rewrite $E - d(N \setminus S)$ as $E - D + d(S)$ and for disjoint S, T we then need to prove, using the fact that $D = d(N)$ and $E \leq D$, that

  $$(E - D + d(S))_+ + (E - D + d(T))_+ \leq (E - D + d(T) + d(S))_+.$$

  Simple case analysis and a geometric intuition is enough.

- Every superadditive game is 0-monotonic, that is for every $S \subseteq T$, it holds that

  $$v(S) + \sum_{i \in T \setminus S} v(i) \leq v(T).$$

  This can be shown by repeatedly using the superadditivity property on $S \setminus T$.

- It is a well-known (but technical to prove) result that for 0-monotonic games, the prekernel coincides with the kernel.

- So far, we have that x is in the prekernel of v.
- Pick any 2-person coalition S, by Lemma L3, $x \restriction S$ is the standard solution of $v^{S,x}$.
- Thanks to Lemma L1 also of $v_{x(S),d \restriction S}$.
- And finally thanks to Lemma L4, $x \restriction S$ is the CG-solution of $(x(S), d \restriction S)$.
- Since we could pick any coalition of size 2, we have that x is the consistent solution of (E, d) by the definition of consistent solution. □

Finally, nucleolus is always in kernel (we skip the proof of that here). This concludes Theorem D.

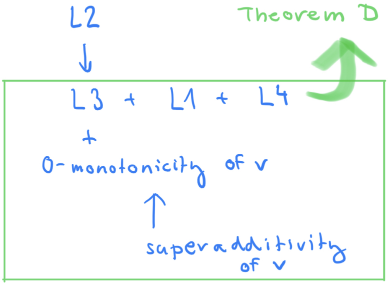

Figure: Scheme of the proof of Theorem D.

Questions