bankruptcy problem

- Where?
  - Game Theory Seminar — led by prof. Martin Loebl
  - webpage — https://kam.mff.cuni.cz/~cerny/teach/seminar.html
  - Zoom
- Who?
  - Jan Bok
    - bok@iuuk.mff.cuni.cz
    - https://iuuk.mff.cuni.cz/~bok/
- When and what?
  - Tuesday 8th March 2021, 9.00 — introduction and history + how to deal with bankruptcy situations
  - Tuesday 15th March 2021, 9.00 — continuation of the previous: properties and historical perspective
  - Tuesday 22nd March 2021, 9.00 — further views on the matter, namely by the lens of cooperative games

problem

- We have a remaining estate of a debtor of value E.
- And a list of claims towards the owner of E.
- fundamental question — **How to divide E between claimants (creditors)?**
- fundamental case — **What if E is smaller than the sum of the claims?**
- possible (simple) approaches
  - divide E proportionally to the size of claims
  - divide E equally between all claimants

history

- Talmud
  - warning — I do not consider myself as an expert on Jewish texts at all. Please, be tolerant regarding any mistakes I have made here. Corrections are welcome!
  - Talmud — Mishnah + discussions, the so called The Gemara
  - Mishnah

    The Mishnah or Mishna (/ˈmɪʃnə/; Hebrew: מִשְׁנָה, "study by repetition", from the verb shanah שנה, or "to study and review", also "secondary") is the first major written collection of the Jewish oral traditions known as the Oral Torah. It is also the first major work of rabbinic literature.
    - The Mishnah was redacted by Judah ha-Nasi at the beginning of the third century CE[4] in a time when, according to the Talmud, the persecution of the Jews and the passage of time raised the possibility that the details of the oral traditions of the Pharisees from the Second Temple period (536 BCE – 70 CE) would be forgotten.
    - The word Mishnah can also indicate a single paragraph of the work, i.e. the smallest unit of structure in the Mishnah. For this reason the whole work is sometimes referred to in the plural form, Mishnayot.
  - two Talmuds
    - they differ mainly in The Gemara
    - Palestinian — created in the area of comtemporary Israel
    - Babylonian
      - newer
      - written in the area of Mesopotamia in the local diaspora
      - It has its own Gemara, often used to help explain the Gemara in Palestinian Talmud.

TABLE I

Estate	Debt 100	200	300
100	33⅓	33⅓	33⅓
200	50	75	75
300	50	100	150

Babylonský Talmud, Kethubot 93 a

There are three creditors; the debts are 100, 200, and 300. Three cases are considered, corresponding to estates of 100, 200, and 300. The *Mishna*[2] stipulates the divisions shown in Table I.

a mysterious table and its caption from Babylonian Talmud

  - background for the situation in the table — description of distribution of estate left by a deceased husband to his wives
    - It was a custom to bind before marriage to leave a certain amount of wealth in case of husband's death.
    - And yes, it was allowed to have more than one wife.
- possible explanations
  - the first case seems to be applying the following rule: Everone gets the same.
  - the third case seems to be applying the following rule: Distribute the estate proportionally to the claims.
  - **but what about the second case?**
    - **So misunderstood, that it led to discussions running for 1700 years!**
    - **Some scholars considered it as a mistake done during translation or a printing mistake.**

paper

JOURNAL OF ECONOMIC THEORY 36, 195–213 (1985)

Game Theoretic Analysis of a Bankruptcy Problem from the Talmud*

ROBERT J. AUMANN AND MICHAEL MASCHLER

The Hebrew University, 91904 Jerusalem, Israel

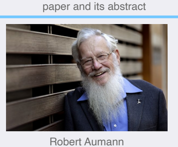
Received August 27, 1984; revised February 4, 1985

DEDICATED TO THE MEMORY OF SHLOMO AUMANN, TALMUDIC SCHOLAR AND MAN OF THE WORLD, KILLED IN ACTION NEAR KHUSH-E-DNEIBA, LEBANON, ON THE EVE OF THE NINETEENTH OF SIVAN, 5742 (JUNE 9, 1982).

For three different bankruptcy problems, the 2000-year old Babylonian Talmud prescribes solutions that equal precisely the nucleoli of the corresponding coalitional games. A rationale for these solutions that is independent of game theory is given in terms of the Talmudic principle of equal division of the contested amount; this rationale leads to a unique solution for all bankruptcy problems, which always coincides with the nucleolus. Two other rationales for the same rule are suggested, in terms of other Talmudic principles. (Needless to say, the rule in question is not proportional division). *Journal of Economic Literature Classification Numbers*: 022, 026, 031, 043, 213. © 1985 Academic Press, Inc.

paper and its abstract

- authors



  - Robert Aumann
    - Nobel Memorial Prize in Economics, John von Neumann Theory Prize, Harvey Prize in Science and Technology, Israel Prize for Economical Research
    - one of the central figures of the field of game theory

  - Michael Maschler — also an influential Israel mathematician
- fundamental assumption — Distribution has to be driven by some logical rules, by some procedure — it is not a nonsense as some Jewish scholars say.
- fundamental questions
  - **How?** — **What is the procedure to obtain the results from the table?**
  - **Why?** — **What led our ancestors to this answer?**
  - With a help of Aumann and Maschler, we shall try to figure out answers to both of these questions.

CG

contested garment - CG

- origin — another Mishnah in Talmud
- story
  - Two older men are arguing over a garment at a market...
  - 1st says: „It is all mine!“
  - 2nd says: „Mine is half of it!“
- resolution?
  - The second de facto concedes the one half to the first.
  - The conceded half is given to the first man.
  - We have the other half in „bank“. — That half is “contested”, there is a fight about it.
  - The remaining, contested half will be divided equally.

	claims 1/2	1
garment 1	1/4	3/4

to sum up, the preceding in a table

- the second view
  - The cumulative claim is one and the half garment.
  - But we have just one garment!
  - The total loss (or we can say overhead) will be equally distributed.
  - We do not distribute what is left, we distribute losses.

Let us transpose this principle to the 2-creditor bankruptcy problem with estate E and claims d_1, d_2. The amount that each claimant i concedes to the other claimant j is $(E-d_i)_+$, where

$$\theta_+ = \max(\theta, 0).$$

The amount at issue is therefore

$$E-(E-d_1)_+-(E-d_2)_+;$$

it is shared equally between the two claimants, and, in addition, each claimant receives the amount conceded to her by the other one. Thus the total amount awarded to i is

$$x_i = \frac{E-(E-d_1)_+-(E-d_2)_+}{2} + (E-d_j)_+. \quad (2.1)$$

We will say that this division (of E for claims d_1, d_2) is prescribed by the CG (contested garment) principle.[7,8]

formally: CG (contested garment) principle

connection to cooperative games - teaser for the next lecture

One more definition is required. The *standard solution* of a 2-person game v with player set $\{1, 2\}$ is given by

$$x_i = \frac{v(12)-v(1)-v(2)}{2} + v(i). \quad (6.7)$$

Note that this is equivalent to $x_1+x_2=v(12)$, $x_1-x_2=v(1)-v(2)$. In words, the standard solution gives each player i the amount $v(i)$ that he can assure himself, and divides the remainder equally between the two players. The nucleolus and kernel,[25] the pre-kernel and the Shapley value of a 2-person game all coincide with its standard solution;

- another view

If one views the solution as a function of E, one obtains the following process. Let $d_1 \leqslant d_2$. When E is small, it is divided equally. This continues until each claimant has received $d_1/2$. Each additional dollar goes to the greater claimant, until each claimant has received all but $d_1/2$ of her claim. Beyond that, each additional dollar is again divided equally. Note that the principle is *monotonic*, in the sense that for fixed claims d_1 and d_2, each of the two awards is a monotonic function of the estate E.

  - draw?
- differences with bankruptcy situations
  - The situation is not clear now, not transparent, everyone “just claims something”.
  - This is why some rabbis disputed relevance of contested garment Mishnah to the bankruptcy problem.
  - **We will show that the CG problem is not only relevant, but it is crucial key towards understanding bankruptcies.**

bankruptcy problem formally

A *bankruptcy problem* is defined as a pair $(E; d)$, where $d=(d_1,..., d_n)$, $0 \leqslant d_1 \leqslant \cdots \leqslant d_n$ and $0 \leqslant E \leqslant d_1 + \cdots + d_n$. A *solution* to such a problem is an n-tuple $x=(x_1,..., x_n)$ of real numbers with

$$x_1 + \cdots + x_n = E$$

(x_i is the amount assigned to claimant i). A solution is called *CG-consistent*, or simply *consistent*, if for all $i \neq j$, the division of $x_i + x_j$ prescribed by the contested garment principle for claims d_i, d_j is (x_i, x_j).

Intuitively, a solution is consistent if any two claimants i, j use the contested garment principle to divide between them the total amount $x_i + x_j$ awarded to them by the solution. It may be verified that the solutions in Table I are consistent.

definition

- the table is satisfying this — show for one of the cases — if there is time
- Theorem A — **Every bankruptcy situation has precisely one consistent solution.**
  - at most one consistent solution

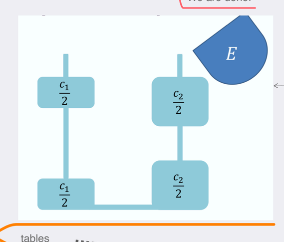
For a contradiction, two different consistent solutions x, y.
There has to be $i, j \in N$ such that: (without loss of generality)
$$y_i > x_i$$
$$y_j < x_j$$
Furthermore, $y_i + y_j \geq x_i + x_j$. *
Remember, if there would be equality, the CG is applied since x and y are consistent. Then they cannot be different!
Now, j gets x_j when dividing $x_i + x_j$ according to CG rule. And y_j when dividing $y_i + y_j$. CG rule is monotonic, so thanks to * $y_j \geq x_j$ ↯ □

  - proof
    - procedure for deriving solution
      - We consider fixed sorted vector of claims and the solution as the value of function taking E as its input.
      - description of procedure

If E is small (i.e. $E \leq n \cdot d_1/2$) we shall distribute E equally.
If 1 reaches $d_1/2$, we stop giving him more.
We now distribute equally between 2...n.
If 2 attains $d_2/2$, we again stop giving him.
We end the first phase in the moment of everyone having half of its claim.
If $E \geq \sum_i d_i/2$, we begin the second phase. The procedure is mirrored.
We distribute losses:
$$\left(\sum_i d_i/2\right) - E.$$
This done until the loss of 1 is $-d_1/2$...

      - The result is a consistent solution.
        - Focus on some fixed two players, we do for them precisely what we have did in the CG problem.
        - It will be even more clear after we shall see an intuitive physical explanation.
  - We are done.

intuition through hydraulic systems



- tables — hydraulic system and Talmudic solutions

TABLE I

Estate	Debt 100	200	300
100	33⅓	33⅓	33⅓
200	50	75	75
300	50	100	150

Babylonský Talmud, Kethubot 93 a

There are three creditors; the debts are 100, 200, and 300. Three cases are considered, corresponding to estates of 100, 200, and 300. The *Mishna*[2] stipulates the divisions shown in Table I.

the mysterious table

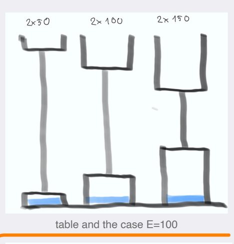


table and the case E=100

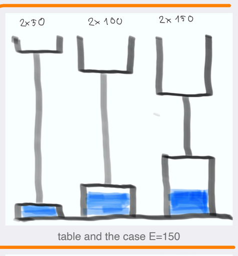
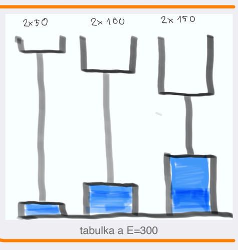


table and the case E=150



tabulka a E=300

self-consistency

Define a *rule* as a function that assigns a solution to each bankruptcy problem. The *CG-consistent* (or simply *consistent*) rule is the one that assigns the CG-consistent solution to each bankruptcy problem. A rule f is called *self-consistent* if

$$f(E; d) = x \quad \text{implies} \quad f(x(S); d|S) = x|S$$

for each set S of creditors, where $x|S$ means "x restricted to S," and $x(S)$ is short for $\sum_{i \in S} x_i$. In words, *any* subset S of the set of all creditors (not only a 2-person subset) uses the rule f to divide among its members the total amount $x(S)$ that it gets when the rule f is applied to the original bankruptcy problem.

definition — The CG-consistent rule is the one described above.

- In other words, the rule (function) distributes in a same way no matter what subset of claimants we choose and the sum of what they got will be the new estate E
- rule vs. solution
  - self-consistency applies to rules
  - consistency to solutions

COROLLARY 3.1. *The CG-consistent rule is self-consistent.*

Proof. Let $(E; d)$ be a bankruptcy problem, x its CG-consistent solution, g the function that assigns to each 2-person bankruptcy problem its CG solution, and S a set of creditors. For *any* i, j, the CG-consistency of x yields $(x_i, x_j) = g(x_i + x_j; d_i, d_j)$; in particular, this is so for $i, j \in S$. But that means that $x|S$ is the CG-consistent solution of $(x(S); d|S)$. Q.E.D.

- The procedure presented above is not necessarily the only self-consistent rule.
  - proportional division
  - equal division

duality and self-duality

- We can think of the procedure as of „cumulative loss distribution“ — Simply reverse the hydraulic system and let the total loss (overhead) pour in!

Define the *dual* f^* of a rule f by

$$f^*(E; d) := d - f(D-E; d);$$

f^* assigns awards in the same way that f assigns losses. A *self-dual* rule is one with $f^* = f$; such a rule treats losses and awards in the same way.

definition

- The CG-consistent rule is self-dual.
  - But it is not a unique self-dual rule. — Proportional distribution.

further (important) properties of the rule and of consistent solution

- individual monotonicity
  - The one with a claim d' will get at least as much as the one with a claim smaller than or equal to d'. Also, higher the claim, higher the losses.
  - also called order-preserving — see formal definition
  - definition

Call a solution x to a bankruptcy problem $(E; d)$ *order-preserving* if

$$0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \quad \text{and} \quad 0 \leqslant d_1 - x_1 \leqslant \cdots \leqslant d_n - x_n.$$

That is, a person with a higher claim than another gets an award that is no smaller and suffers a loss that is no smaller;

- global monotonicity
  - By increasing E (and fixing claims), the individual gains will be at least as big as before the increase of E.
  - a property of rule
  - intuitively — We pour more liquid into the hydraulic system.
- symmetry
  - Any two claimants with the same claims get the same.
  - Can be stated as a property of both rule and solution.

coalition formation

In discussing our Mishna, the Jerusalem Talmud[22] says as follows: "Samuel says, the Mishna takes it that the creditors empower each other; specifically, that the third empowers the second to deal with the first. She may say to her, 'Your claim is 100, right? Take 50 and go.'"

Samuel is obviously referring to the cases $E=200$ and $E=300$. The second and third creditors (whose claims are 200 and 300, respectively) form a coalition "against" the first (whose claim is 100). This leaves two effective protagonists, with claims of 500 and 100, respectively; applying the contested garment rule yields the first creditor 50, with the remainder going to the coalition. If the coalition again applies the contested garment rule to divide its award among its members, the numbers in the Mishna result.

- again Mishna — The claimants can group together and empower each other. — Let us formalise it and derive an alternative description of the CG-consistent rule.
- coalitional procedure

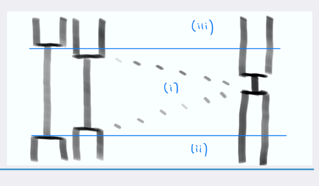
Depending on the values of E and d, we treat a given n-person problem in one of the following three ways:

(i) Divide E between $\{1\}$ and $\{2,..., n\}$ in accordance with the CG solution of the 2-person problem $(E; d_1, d_2 + \cdots + d_n)$, and then use the $(n-1)$-person rule, which we know by induction, to divide the amount assigned to the coalition $\{2,..., n\}$ between its members.

(ii) Assign equal awards to all creditors.

(iii) Assign equal losses to all creditors.

Specifically, (i) is applied whenever $nd_1/2 \leqslant E \leqslant D-(nd_1/2)$. We apply (ii) when $E \leqslant nd_1/2$, and (iii) when $E \geqslant D-(nd_1/2)$. This is called the *coalitional procedure*.



THEOREM C. *The coalitional procedure yields the consistent solution for all bankruptcy problems.*

- proof
- We got an alternative proof to Theorem A.
- We see for the first time coaltions at play.

historical and social aspects

- the half and its meaning in Takmud and Jewish teachings
  - principle “more than half is like the whole“
    - the reason for the qualitative change at E=D/2
    - We aim on fulfilling at least the half of each claim first. — Observe through the hydraulic intuition.
  - an example of loan
    - again a Mishnah
    - Rachel borrowed some money to Isaac.
    - Isaac goes bankrupt --- he is insolvent.
    - But he has a house.
    - If the house is valued to more than the half of Rachels claim, she has the right to take the house.
    - If the house has value lesser than the half of the Rachel's claim, the loan was given based on friendship or a good word or it was not wise of Rachel to provide it at all. Isaac retains the house.
- „social justice“
  - Why those quotation marks? — The concept of social justice is subjective (and delicate).
    - Everyone has different views and opinions.
    - A need for consensus.
    - **What it means to be fair?**
  - Remember the axioms and properties mentioned earlier.
  - Observe the approach to dividing the estate. — First, we try to satisfy all claims up to 50 %, so that everyone has „basically everything“ and not „basically nothing“.

conclusion

- We have provided two different viewpoints on how to derive the results from the mysterious Talmud table. (The third one will be in the next lecture.) — That is, we gave an answer to „Why?“
- It is clear that the people who introduced the solutions to bankruptcy problems did not know game theory.
  - Nevertheless, we have shown that there are clear logical and natural rules and a procedure yielding the historic solutions.
  - However, these rules have not been written down or in any way preserved...
  - ...and we have been left just with examples.
- final thought (at least for today) — No method should be described solely on an example basis. 😁
- next Tuesday
  - introduction to cooperative game theory — mainly focused on fundamental solution concepts („stability points“)
  - How can we state the results on bankruptcy problems in the framework of cooperative games? Why it is useful? What surprises are hidden in it? — We shall see that cooperative game theory is a powerful tool to model decision-making problems and also an expressive framework for such situations.
- questions?