Topics for Cooperative Game Theory Seminar

September 30, 2024

Cooperative games - recap

Definition 1. *A cooperative game is* (N, v) *where* $N = \{1, \dots, n\}$ *is the player set and* $v \colon 2^N \to \mathbb{R}$ *with* $v(\emptyset) = 0$ *is the characteristic function of the game.*

The goal of cooperative game theory is to find *solutions* to cooperative games, namely *How to distribute* $v(N)$ *among the agents?*

Definition 2. *A solution of a cooperative game* (N, v) *is* $\Sigma(v) \subseteq \mathbb{R}^n$.

There are two most studied solution for cooperative games, one focusing on *stability* and the other on *fairness* of the division.

Definition 3. *For a cooperative game* (N, v)*, the* core $\mathcal{C}(v)$ *is defined as*

$$\mathcal{C}(v) = \{x \in \mathbb{R}^n \mid x(S) \geq v(S), \forall S \subseteq N \text{ and } x(N) = v(N)\}. \tag{1}$$

Definition 4. *For a cooperative game* (N, v)*, the* Shapley value $\phi(v) \in \mathbb{R}^n$ *is defined for every* $i \in N$ *as*

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \gamma_S \left[v(S \cup i) - v(S) \right]. \tag{2}$$

1 Set Functions Overview

The characteristic function of a cooperative game $v \colon 2^N \to \mathbb{R}$ is a *set function*. The set functions are studied not only in regard to cooperative game theory but also in other areas such as combinatorial optimization, auctions, assignment problems, capacities and measure theory, operation research, and machine learning.

This seminar will give an introductory overview of the application of set functions in these areas and some of the problems studied regarding set functions.

2 Bubbles - combinatorial problem in games with restricted cooperation

A cooperative game with restricted cooperation is defined as $(N, v, \mathcal{F})$, where $\mathcal{F} \subseteq 2^N$ are the *feasible sets* and $v \colon \mathcal{F} \to \mathbb{R}$ is the restricted characteristic function. While characterising the Shapley value in this setting, there is a problem, which we call the bubbles. It is defined for games with restricted cooperation, with $\mathcal{F}$ following a specific structure.

Definition 5. *A set system* $\mathcal{F} \subseteq 2^N$ *is* AUS (Accessible union closed system *if it satisfies*

1. *(Accesibility)* $\forall \emptyset \neq S \in \mathcal{F}, \exists i \in S : S \setminus i \in \mathcal{F}$,
2. *(Weak union closedness)* $\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset : S \cup T \in \mathcal{F}$.

The problem is now as follows:

Problem 1. *Is there* $\forall i \in N$ *an agent* $j \in N \setminus i$ *and a set* $\mathcal{S} \subseteq \mathcal{F}$ *satisfying*

1. $\forall S \in \mathcal{S} : i, j \in S$,
2. $\mathcal{F} \setminus \mathcal{S}$ *is AUS.*

3 Opinion Difusion - Where is the Shapley value?

At the Spring school, Lluís presented his topic on Opinion Difusion. Simply stating, there is a social network in which the vertices correspond to agents who influence opinions of their neighbours. The theory then studies questions connected to converge of opinion distribution.

The idea is to look for connections between cooperative game theory and opinion difusion problem.

4 Sequential truth learning

This problem is from the area of social learning. There is an initial signal of a truth, which each agent learns with a certain probability and based on the decision of the agents before him, the agent has to decide what is his belief about the initial signal.

Filip can tell us about a project he has been working on while on a research stay in Rutgers University.

5 Parameter playing

Assume there are parameters $\theta_1, \ldots, \theta_m \in \mathbb{R}$, which define a cooperative game (N, v_Θ) in a unique way. Now assume the following problem.

Problem 2. *How many values of (N, v_Θ) is in general necessary for learning the parameters $\theta_1, \ldots, \theta_m$?*

Examples for some of the known games lead to interesting connections to linear algebra or the sorting problem.

The plan for the seminar is to generalize these results.

6 Discrete cooperative games

The model of cooperative games does not immediately apply to situations with indivisible goods, as the solution concepts such as the core or the Shapley value do not result in integer solution even for an integer characteristic function $v \colon 2^N \to \mathbb{Z}$.

While working on a bachelor thesis of Richard Mužík, we came to different modifications of the Shapley value, which might be interesting to discuss and extend the result here.

7 Jan Bok problem

Jan Bok was working on a side project in incomplete cooperative games, which was connected to combinatoral problems. We are eager to her about this research of his, with hopes that this will motivate him to proceed in the research and possibly open new problems to work on together.

8 David Hartman problem

David Hartman will present one of the many suggestions he told me about during our random corridor meetings.

References