Interval games

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Motivations and introduction

Cooperative game

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- $v(S)$ ... values of coalition
- usually $N = \{1, \ldots, n\}$

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  - $x(S) := \sum_{i \in S} x_i$
- $\mathcal{I}(v) = \{x \in \mathcal{I}^*(v) \mid \forall i \in N : x_i \geq v(i)\}$ ... **imputation**

Idea: *Payoff distribution leads to cooperation...*

The core

For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\}.$$

- $v(N)$ ... value, which is distributed among players
- $x(S) > v(S) \implies$ coalition S does not leave N

- **monotonic game** $(S \subseteq T \subseteq N)$

$$v(S) \leq v(T)$$

- **superadditive game** $(S, T \subseteq N, S \cap T = \emptyset)$

$$v(S) + v(T) \leq v(S \cup T)$$

- **convex game** $(S, T \subseteq N)$

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$$

- **Definition (Interval):** An interval X is a set
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  with $\underline{X}$ being the lower bound and $\bar{X}$ being the upper bound of the interval.

Background: Interval analysis

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- By interval we mean closed interval
- We denote set of real intervals by $\mathbb{IR}$

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  - $X * Y := [min(S), max(S)], S = \{\underline{X}\bar{Y}, \bar{X}\underline{Y}, \underline{X}\underline{Y}, \bar{X}\bar{Y}\}$

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  - $X/Z := [min(S), max(S)], S = \{\underline{X}/\bar{Z}, \bar{X}/\underline{Z}, \underline{X}/\underline{Z}, \bar{X}/\bar{Z}\}$

Cooperative interval game

A Cooperative interval game is an ordered pair (N, w), where $N = \{1, 2, ..., n\}$ is a set of players and $w : 2^N \to \mathbb{IR}$ is a characteristic function of the cooperative game. We further assume that $w(\emptyset) = [0, 0]$.

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- The set of all interval cooperative games on a player set N is denoted by $IG^{|N|}$

border games

For every $(N, w) \in \mathbb{N}$, border games $(N, \underline{w}) \in G^N$ (lower border game) and $(N, \bar{w}) \in G^{|N|}$ (upper border game) are given by $\underline{w}(S) = \underline{w(S)}$ and $\bar{w}(S) = \overline{w(S)}$ for every $S \in 2^N$

1st approach

Weakly better operator

Interval I is weakly better than interval J ($J \succeq I$) if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$.

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- Set of all interval imputations of $(N, w) \in G^N$:
  - $\mathcal{I}(w) := \{(I_1, ..., I_{|N|}) \in \mathbb{IR}^{|\mathbb{N}|} \mid \sum_{i \in N} I_i = w(N), I_i \succeq w(i), \forall i \in N\}$

Cooperative interval games: 2 approaches

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- Set of interval selection core of $(N, w) \in G^N$:
  - $\mathcal{C}(w) := \{(I_1, ..., I_{|N|}) \in \mathcal{I}(w) | \sum_{i \in S} I_i \succeq w(S), \forall S \in 2^N \setminus \emptyset\}$

2nd approach

Selection

A game $(N, v) \in G^N$ is a selection of $(N, w) \in IG^N$ if for every $S \subseteq N$ we have $v(S) \in w(S)$. Set of all selections of (N, w) is denoted by $\mathrm{Sel}(w)$

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- Set of all interval selection imputations of $(N, w) \in IG^N$:
  - $\mathcal{SL}(w) = \bigcup\{\mathcal{I}(v) | v \in Sel(w)\}$
- Set of interval selection core of $(N, w) \in IG^N$:
  - $\mathcal{CL}(w) = \bigcup\{\mathcal{C}(v) | v \in Sel(w)\}$

Selection monotonic interval game

An interval game (N, w) is selection monotonic if all its selections are monotonic games. The class of such games on set of N players is denoted by $SeMIG^N$

Selection superadditive interval game

An interval game (N, w) is selection superadditive if all its selections are superadditive games. The class of such games on set of N players is denoted by $SeSIG^N$

Selection convex interval game

An interval game (N, w) is selection convex if all its selections are convex games. The class of such games on set of N players is denoted by $SeCIG^N$

Theorem 1.

An interval game (N, w) is selection monotonic if and only if for every $S, T \in 2^N, S \subset T$

$$\bar{w}(S) \leq \underline{w}(T).$$

Proof: $\rightarrow$:

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- Suppose that (N, w) is selection monotonic and $\bar{w}(S) > \underline{w}(T)$ for some $S, T \subseteq N$, where $S \subset T$.
- (N, v) with $v(S) = \bar{w}(S)$ and $v(T) = \bar{w}(T)$ clearly violates monotonicity

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Proof: $\leftarrow$:

- Suppose $S, T \subseteq N$ and WLOG $S \subset T$
- Monotonicity cannot be violated since $v(S) \leq \bar{w}(S) \leq \underline{w}(T) \leq v(T)$.

Theorem 2

An interval game (N, w) is selection superadditive if and only if for every $S, T \in 2^N, S \cap T = \emptyset, S \neq \emptyset, T \neq \emptyset$

$$\bar{w}(S) + \bar{w}(T) \leq \underline{w}(T \cup S).$$

Theorem 3

An interval game (N, w) is selection convex if and only if for every $S, T \in 2^N, S \cap T = \emptyset, S \neq \emptyset, T \neq \emptyset, S \not\subseteq T, T \not\subseteq S$

$$\bar{w}(S) + \bar{w}(T) \leq \underline{w}(T \cup S) + \underline{w}(T \cap S).$$

proof of both theorems is similar to proof of theorem 1, so I'll leave it as an excercise to the listeners

Core coincidence

Coincidence problem

Under which conditions core of cooperative game coincides with core of the game in terms of selections of the interval game ?

The function $gen : 2^{\mathbb{IR}^N} \rightarrow 2^{\mathbb{R}^N}$ maps to every set of interval vectors a set of real vectors. It is defined as:

$$gen(S) = \bigcup_{s \in S} \{(x_1, x_2, ..., x_n) | x_i \in s_i\}$$

Reformulation of problem

What are the neccecary and sufficient conditions to satisfy $gen(\mathcal{C}(w) = \mathcal{SC}(w))$?

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- For any $x \in gen(\mathcal{C}(w))$, the $\underline{w}(N) \leq \sum_{i \in N} x_i \leq \bar{w}(N)$ obvious.
- x is in the core fory any selection of the interval game (N, s) given by:
  - $$s(S) = \begin{cases} [\sum_{i \in N} x_i, \sum_{i \in N} x_i] & \text{if } S = N \\ [\underline{w}(S), min(\sum_{i \in S} x_i, \bar{w}(S)] & \text{otherwise.} \end{cases}$$

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- Clearly, $Sel(s) \subseteq Sel(w)$ and $Sel(s) \neq \emptyset$. Therfore $gen(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$.

Core coincidence characterisation

For every interval game (N, w) we have $gen(\mathcal{C}(w)) = \mathcal{SC}(w)$ if and only if for every $x \in \mathcal{SC}(w)$ there exist non-negative vectors $l^{(x)}$ and $u^{(x)}$ such that:

1. $\sum_{i \in N}(x_i - l_i^{(x)}) = \underline{w}(N)$,
2. $\sum_{i \in N}(x_i + u_i^{(x)}) = \overline{w}(N)$,
3. $\sum_{i \in S}(x_i - l_i^{(x)}) \geq \underline{w}(S), \forall S \in 2^N \setminus \{\emptyset\}$,
4. $\sum_{i \in S}(x_i + u_i^{(x)}) \geq \overline{w}(S), \forall S \in 2^N \setminus \{\emptyset\}$.

Proof

First, we observe that Theorem 7 taken into account, we only need to take care of $gen(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$ to obtain equality.

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Proof $\mathcal{SC}(w) \subseteq gen(\mathcal{C}(w))$

- suppose we have some $x \in \mathcal{SC}(w)$
- For x, we need to find some interval $X \in \mathcal{C}(w)$ such that $x \in gen(X)$.
- This is equivalent to the task of finding two nonnegative vectors $l^{(x)}$ and $u^{(x)}$ such that:

$$([x_1 - l_1^{(x)}, x_1 + u^{(x)}], [x_2 - l_2^{(x)}, x_2 + u_2^{(x)}], ..., [x_n - l_n^{(x)}, x_n + u_n^{(x)}]) \in \mathcal{C}(w)$$

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- From the definition of interval core, we can see that these two vectors have to satisfy exactly the mixed system 4.1 - 4.4.