Cooperative game theory

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Fairness in the model of cooperative games

What is fair?

What is the most fair payoff distribution?

- we revise already studied solution concepts
- we define new ones
- we learn how to compare them
- we introduce model incorporating player's individual notions of fairness

The Shapley value and the nucleolus

The Shapley value

For a cooperative game (N, v), the Shapley value $\phi(v)$ is defined as

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left[v(S \cup i) - v(S)\right]$$

- is considered as a fair solution (discussed earlier)
- often outside the core

The nucleolus

For a cooperative game (N, v), the **nucleolus** $\eta(v)$

$$\eta(v) := \{x \in \mathcal{I}(v) \mid \theta(y) \succeq_{lex} \theta(x) \text{ for } y \in \mathcal{I}(v)\}.$$

- η is *fair* core selection
- for many games: $\phi(v) \neq \eta(v)$

1. b^v ... **Utopia vector**
   - $b^v_i := v(N) - v(N \setminus i)$
   - *Higher demand is not taken seriously...*
     - $v(N \setminus i) > v(N) - b^v_i$
     - coalition $N \setminus i$ forms
2. a^v ... **Minimal right vector**
   - the world is not utopia: $\sum_{j \in N} b^v_j > v(N)$
   - $a^v_i := \max_{S: i \in S} v(S) - \sum_{j \in S \setminus i} b^v_j$
     2.1 pay players from $S \setminus i$ their utopia value
     2.2 take the rest
   - find the best coalition S for you
     - your minimal right

- for $x \in \mathcal{C}(v)$
  - $a_i^v \leq x_i \leq b_i^v$
- we choose efficient compromise...

The τ-value

The τ**-value** $\tau(v)$ for a cooperative game (N, v) is defined as a convex combination of a^v and b^v satisfying $\sum_{i \in N} \tau(v)_i = v(N)$.

- $a^v(N) \leq v(N) \leq b^v(N)$ holds for *quasibalanced games*

VALUES: ϕ,η, τ

The values **are** fair...:

- ϕ is often considered as a fair solution (discussed earlier)
- η is *fair* core selection
- τ is a **fair** compromise between utopia vector and minimal right vector

...or are they?

- ϕ and τ are often **not** contained in the core
- often: $\phi(v) \neq n(v) \neq \tau(v)$
- Which value should we choose?

"*I will share if I can...*"

Bilateral transfer

Tuple (i, j, α, x) is ***bilateral transfer***, if

$$x_i - \alpha \geq x_j + \alpha.$$

- i, j ... me and you
- $x \in I(v)$ ... what we get
- $\alpha \geq 0$ ... what I share with you

"... but it has to be a stable transfer."

Egalitarian core

Imputation $x \in C(v)$ is ***egalitarian*** if there *does not exist* $y \in C(v)$, which would be a result of a bilateral transfer (i, j, α, x).

"*Whatever you do, this is the best possible outcome...*"

Strong egalitarian core

Imputation $x \in C(v)$ is ***strongly egalitarian*** if there *does not exist* $y \in C(v)$, which would be an outcome of finitely many bilateral transfers.

egalitarian core $\mathcal{C}_E(v)$

- exists, if $C(v) \neq \emptyset$
- multi-point solution concept
- $\mathcal{C}_{SE} \subseteq \mathcal{C}_E$

strongly egalitarian core $\mathcal{C}_{SE}(v)$

- single-point solution concept
- solution of the least squares:
- $\min_{y \in \mathcal{C}(v)} \|y\|_2$

1. **fair** thanks to *bilateral transfers*
2. **rational** thanks to the stability of the *core*

Example

Game of two players (N, v), where $v(1) = 1, v(2) = 0$ a $v(12) = 2$.

$\mathcal{C}_E(v) = \{(1, 1)^T\}$ ... why should 1 cooperate?

$\phi(v) = (1.5, 0.5)^T$ ... this is *more fair*

- One could say: "We *overdo* the fairness..."

Fairness predicates

"Division of solution concepts into elementary properties..."

Definition

A ***predicate on the imputation space*** of a cooperative n-person game is a mapping $\mathcal{P}$ that assigns every game (N, v) a subset $\mathcal{P}(v) \subseteq I(v)$.

Fairness Predicates

- subset of $I(v)$
- **does not** have to *make sense* on itself:
- Dummy player predicate DP
  - rules out $x \in I(v) : x_i > 0$ for i with contribution 0
  - not much of a concept

Solution concept

- subset of $I(v)$ (usually)
- **does** have to *make sense* on itself:
- Shapley value
  - *fair* distribution of payoff given by rules (EFF, ADD, DP, SYM)
  - an interesting concept

"Axioms as predicates..."

A (partial) one-point solution concept $\mathcal{P}$ satisfies

- **anonymity** if for any permutation σ of the player set N we have $\mathcal{P}(v)_i = \mathcal{P}(\sigma(v))_{\sigma(v)}$
- **additivity** if for two cooperative n-person games (N, v) and (N, w) the equation $\mathcal{P}(v + w) = \mathcal{P}(v) + \mathcal{P}(w)$ holds.
  - $\mathcal{P}(v) \neq \emptyset$ and $\mathcal{P}(w) \neq \emptyset$

A predicate $\mathcal{P}$ on the imputation space of cooperative n-person games

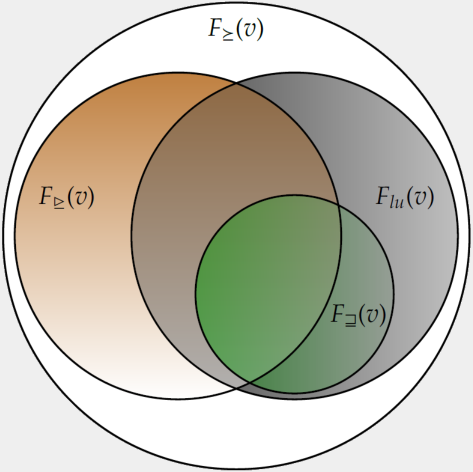
- **split** if for all (N, v) we have $\mathcal{P}(v_0) + s(v) = \mathcal{P}(v)$
  - $s(v)_i = v(i)$

"We are interested if solution concepts satisfy predicates..."

Fairness based on desirability

"If you work hard, you should get more."

4 desirability predicates:



"If you work hard, you should get more."

Definition

Player desirability relation $i \succeq j$ denotes that player i is more desirable than j, i.e.

$$v(A \cup \{i\}) \geq v(A \cup \{j\}) \text{ for } A \subseteq N \setminus i, j.$$

Definition

Player desirability-fair imputation $x \in I(v)$ is such that

$$i \succeq j \implies x_i \geq x_j.$$

The set of all such x is denoted by $F_{\succeq}(v)$.

$F_{\succeq}(v)$ AND SOLUTION CONCEPTS

Theorem

For a game (N, v), following hold.

1. $Ker(v) \subseteq F_{\succeq}(v)$
2. $n(v) \in F_{\succeq}(v)$
3. (N, v) *is quasi-balanced* $\implies$ τ*-value* $\tau(v) \in F_{\succeq}(v)$,
4. (N, v) *super-additive* $\implies$ *Shapley value* $\phi(v) \in F_{\succeq}(v)$,
5. *If* $C(v) \neq \emptyset \implies \emptyset \neq C_E(v) \subseteq F_{\succeq}(v)$.

Open questions:

- What about other solution concepts? (bargaining set, the prekernel, ...)
- What are full characterisations of 3.,4.
- ...

Weak Desirability of players $F_{\trianglerighteq}(v)$

"I don't know if it holds, but I feel like it does..."

desirability: $i \succeq j \implies v(A \cup \{i\}) \geq v(A \cup \{j\})$ for $A \subseteq N \setminus i, j$ $\#$ of

conditions: $2^{|N|-2}$ Problem: infeasible to check for even a

relatively small number of players

- Solution: pick a subset of conditions
  - *individual payoffs* and *marginal contributions* to N

Weak Desirability of players $F_{\trianglerighteq}(v)$

1. *individual payoffs*
    - $v(i) \geq v(j)$
2. *marginal contributions to the grandcoalition N*
    - $v(N) - v(N \setminus i) \geq v(N) - v(N \setminus j)$

Definition

Player weak desirability relation $i \trianglerighteq j$ denotes that player i is more desirable (in a weak sense) than j, i.e.

$$v(i) \geq v(j) \text{ and } v(N \setminus i) \leq v(N \setminus j).$$

Definition

Weak player desirability-fair imputation $x \in I(v)$ is such that

$$i \trianglerighteq j \implies x_i \geq x_j.$$

The set of such x is denoted by $F_{\trianglerighteq}(v)$.

$F_{\trianglerighteq}(v) \subseteq F_{\succeq}(v)$

- $i \trianglerighteq j$ is weaker than $i \succeq j$
- therefore, it is *activated* more often
- $\succeq$ holds for at least as much pairs of players as $\trianglerighteq$
- Example:
  - $i_1 \trianglerighteq i_2, i_3 \trianglerighteq i_4 \implies x_{i_1} \geq x_{i_2}, x_{i_3} \geq x_{i_4}$
  - $i_3 \succeq i_4 \implies x_{i_3} \geq x_{i_4}$
- Consequence: $F_{\trianglerighteq}(v) \subseteq F_{\succeq}(v)$

"Is it interesting? Nobody knows yet..."

Theorem

For a game (N, v), following hold:

1. *(N, v) is 1-convex $\implies \tau(v) \in F_{\trianglerighteq}(v) \cap C(v)$,*
2. *(N, v) is quasi-balanced and a little condition $\implies$ $\tau(v) \in F_{\trianglerighteq}(v)$.*

Open questions:

- **basically the rest!**

Desirability relation on coalitions $F_{\sqsupseteq}(v)$

"United we stand, divided we fall..."

Definition

Desirability relation on coalitions $A \sqsupseteq B$ denotes coalition A is more desirable than B, i.e.

$$v(C \cup A) \geq v(C \cup B) \text{ for all } C \subseteq N \setminus (A \cup B).$$

Definition

Coalition desirability-fair imputation $x \in I(v)$ is such that

$$A \sqsupseteq B \implies x(A) \geq x(B).$$

The set of such x is denoted by $F_{\sqsupseteq}(v)$.

DESIRABILITY RELATION ON COALITIONS $F_{\sqsupseteq}(v)$

"But we actually mostly fall..."

- $i \succeq j \iff \{i\} \sqsupseteq \{j\}$
- $F_{\sqsupseteq}(v) \subseteq F_{\succeq}(v)$
- exists game (N, v):
  - $F_{\sqsupseteq}(v) \cap C(v) = \emptyset$
  - $\tau(v) \notin F_{\sqsupseteq}(v)$
  - $\phi(v) \notin F_{\sqsupseteq}(v)$
  - $n(v) \notin F_{\sqsupseteq}(v)$

- same problem as for $\succeq$:
  - 2^N coalitions
  - many of them *unlikely*
- Task: select a sensible subset of condition
  - coalition of substitutes K (*labor union*)
  - $K \sqsupseteq \{i\}$ (*factory owner* i)
  - $x(K) \geq x_i$ (K: "We are not slaves!")

Definition

The labor union-fair imputation $x \in I(v)$ is such that

1. $K \sqsupseteq \{i\} \implies x(K) \geq x_i$,
2. $x \in F_{\succeq}(v)$.

The set of such x is denoted by $F_{lu}(v)$.

"At least the egalitarian core C_e is fair for the workers."

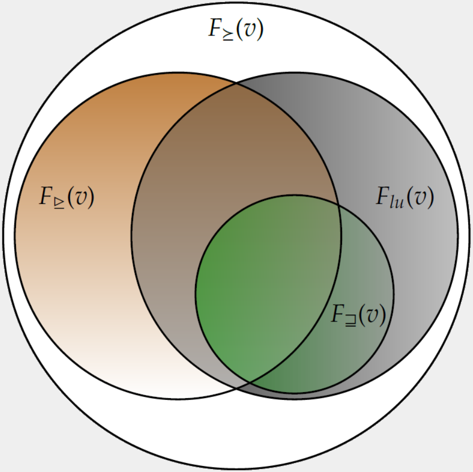
Theorem

$C_e \subseteq F_{lu}(v)$ for convex games (N, v).

Also, minor results about Shapley, τ-value and nucleolus.

"If you work hard, you should get more."

4 desirability predicates:



"This is fair, and that is fair, so which one is more fair?"

1. Is the fairness predicate actually a good one?
   - In general, it might be empty for a game (N, v)
   - For a special case: Always *better* solution than other
     - $F_{\sqsupseteq}(v)$ for banktruptcy games non-empty
     - otherwise hard to say
2. which fairness predicate is *better*?
3. we can find *unpleasent* games for the specific predicate
   - Do these games really matter?

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Definition

A predicate $\mathcal{P}$ is **satisfiable within the core** (in a class G) if

$$(N, v) \in G : C(v) \neq \emptyset \implies \mathcal{P}(v) \cap C(v) \neq \emptyset.$$

We say $\mathcal{P}$ is *core-satisfiable* or simply *satisfiable*.

"It is good, at least when the game is stable."

Definition

A predicate $\mathcal{P}$ is **satisfiable within the core** (in a class G) if

$$(N, v) \in G : C(v) \neq \emptyset \implies P(v) \cap C(v) \neq \emptyset.$$

- we can define different ?-satifiability
- Core-satisfiability enoforces stability of the solution

"And how does it look, from the core point-of-view?"

Theorem

1. *$F_{\succeq}(v)$ is satisfiable for every game,*
2. *$F^{o}_{\succeq}(v)$ is satisfiable for every game,*
3. *$F_{\trianglerighteq}$ is satisfiable for every convex and 1-convex game,*
4. *$F_{\trianglerighteq}$ is **not** satisfiable for every superadditive game,*
5. *F_{lu} is satisfiable for every convex game, but **not** every superadditive game.*

Individual or Culture Specific Notions of Fairness

"This is fair to you?"

- the most natural setting
  - not only different interests
  - but also notions of fairness
- modification in the *stability* notion (different from Core)

"The core sounds fine, but lets keep it sensible..."

imputation $x \in C(v)$ if

- $x(S) \geq v(S)$
- if S does not form (does not agree on fair notion)
  1. why should we consider this condition?
     - why shouldn't we allow for $y \notin C(v)$?
  2. why should we agree on x?
     - our differences might block all $x \in C(v)$
- my fairness notion $=$ my **culture** (cultural identification)
- How does our cultural differences affect us?

Modified stability condition

"To work together, we have to find a common ground."

- F_i ... fairness predicate (**Cultural identification of player i**)
- $F_i(w)$ ... acceptable imputations of i in (N, w)
  - imputation outside $F_i(w)$ results in **no** cooperation

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A coalitions S is **culturally compatible** (in a game (N, v)) if either

1. $S = \{i\}$
2. exists $x \in \cap_{i \in S} F_i(v_S)$:
   2.1 $x(S) = v_S(S)$
   2.2 $x(A) \geq v_S(A)$ for every $A \subseteq S$ **culturally compatible**

- F_i ... fairness predicate (**Cultural identification of player i**)
- $F_i(w)$ ... acceptable imputations of i in (N, w)
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   2.2 $x(A) \geq v_S(A)$ for every $A \subseteq S$ **culturally compatible**

Culturally compatible core

Let (N, v) be a cooperative game and let $CC(v)$ be the set of its culturally compatible coalitions.
A **culturally compatible core** C_{cc} is

$$C_{cc}(v) = \{x \in \cap_{i \in N} F_i(v) \big| x(N) = v(N) \text{ and } x(A) \geq v(A), \forall A \in CC(v)\}.$$

- any such x