Cooperative game theory

David Ryzák

Lecture website: kam.mff.cuni.cz/∼cerny

david.ryzak99@gmail.com

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Outline

Stochastic cooperative games

Motivation

- Framework of only one game (not dynamic).
- The main question is how to divide payoff of the grand coalition.
- When to use stochastic model?
    1. Information shortage (Costly to get all the information or not possible to get it).
    2. Decision before the realization of random event (demand, events in principle random) .

- We can use classical framework of cooperative game theory (N, v) with some additions and generalizations.
- Characteristic function is considered to be random variable.

Random characteristic function

$v(S) : \Omega \longrightarrow E,\ S \subseteq N$
Ω possible outcomes of random variable
E measurable space (for us mostly $\mathbb{R}$)

Models for V

- Normal distribution $\mathcal{N}(\mu, \sigma)$ or $\mathcal{N}_{2^N}(\mu, \Sigma)$
  e.g., Model for random erorr.
- Uniform distribution on a set.
  Mostly intervals $[a, b]$.
  e.g., We do not know anything about the value, but we the bounds.
- Discrete distribution.
  e.g., Information about number events which can happen with positive probability.

Example of storage

Two sellers want to decide whether they want to share their cost of storing the product.
Random part is a demand of their product.

Bankruptcy situation

Two sellers want to decide whether they want to share their cost of storing the product.
Random part is a demand of their product.

Question

Can we already make a decision about the division of players costs or payoffs in case of (N, v), where v is random?

Answer

Yes, but let us look at model with one more component.

Stochastic models II

Model with preferences

Triple $(N, v, (\preceq_i)_{i \in N})$ is the stochastic model with preferences, where $(\preceq_i)_{i \in N}$ is a preference of a player over random variables and v is random.[1] or [2]

Preferences

Player i prefers random variable X over Y if $Y \preceq_i X$.

- There is more model with preferences. Their difference are in the expression of the allocation.

Model with preferences I

Solution is in the form of $x_i = d_i + r_i \cdot (v(N) - \mathbb{E}(v(N)))$,

- $\sum_{i \in N} r_i = 1$,
- $r_i \geq 0$,
- $\sum d_i = \mathbb{E}(v(N))$

Model with preferences II

Solution is in the form of $x_i = r_i \cdot v(N)$.

Core in model with preferences I

- In the paper they use mostly quantiles $u^X_{\alpha_i}$ as preferences are used for a given $\alpha_i \in (0, 1)$.

Core of the stochastic cooperative game $(N, v, (\preceq_i)_{i\in N})$

Coalition $S \subset N$ has no incentive to split off if and only if
$\sum_{i\in S} \left(d_i + r_i(u^{v(N)}_{\alpha_i} - \mathbb{E}(v(N)) \right) \geq max_{i\in S} u^{v(S)}_{\alpha_i}$

Example of the model core with preferences

Bankruptcy problem

On the whiteboard, two player bankruptcy game.

Properties of stochastic coop game

- Some results can be generalized for the stochastic approach.

Convex games in model with preferences

The greater the coalition the greater the marginal contribution of the coalition or player.
In stochastic setting is game convex if:
$d_i^{T \cup U} + r_i^{T \cup U} v(T \cup U) \succeq_i d_i^{S \cup U} + r_i^{S \cup U} v(S \cup U), \forall i \in U$ and
$d_i^{T \cup U} + r_i^{T \cup U} v(T \cup U) \succeq_i d_i^{T} + r_i^{T} v(T), \forall i \in T$

Properties of stochastic coop game

Convex games in model with preferences

Convex stochastic cooperative game $(N, v, \preceq)$ has nonempty core.

Preferences in the model

- Quantiles as we used.
- Stochastic dominance of some order(first, second)
- $\mathbb{E}(X) + b\sqrt{Var(X)}$ (Similar condition for core can be derived in the case of these preferences)

Model with preferences II

Solution is in the form of $x_i = r_i \cdot v(N)$.

- In this model we are able to quite nicely define Shapley value.
- The portion of the grand coalition payoff.
- Shapley value can be defined in a few equivalent ways in the deterministic setting.
- In stochastic setting solutions does not coincide. More in [2].

Other types of models: Based on optimization of a function

- Another variant is to use specify multivariate distribution.
- The model is somehow generalizing the previous models: allocation $x_i = d_i + r_i(v(N) - \mathbb{E}(v(N)))$, where $r_i \in \mathbb{R}$.
- Based on optimizing given function [3].

Objetive function

Minimize $\sum_{S\subset N} \mathbb{E}(e(S,x) - \hat{e}(v,x))$, where

- $\hat{e}(v,x)) = \frac{1}{2^n-1} \sum_{S\subset N} e(S,x)$
- and $e(S,x) = v(S) - x(S)$

We get explicit expression for d_i and r_i for each player.

Conclusion

We went through the some of the possible models to deal with randomness in the model for cooperative games. (There are other possibilities how to deal with uncertainty like model with states or scenarios and fuzzy models).

Reference I

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