Algorithmic fairness

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- What is fair?
- Fair is subjective
- Activities with a sole purpose to harm others clearly in our psychology
- We make assumptions:

- ethical = rational
- define fairness
- optimize fairness

Will present some introductory examples

- winter road maintenance: joint work with Fink, Pelikánová, Tancer
- Allocation with connected bundles
- Critical distribution system: on-going work with large team
- Median of Partitions (1950's)
  (L, Vegh, Sereni)

Arc-routing for winter road maintenance

Maintenance Plan

Given $G=(V,E)$, $D \subseteq V$

- Partition E: $(P_1, \dots, P_r)$; assign $D \ni d_i$ to P_i
- $P_i \subseteq R_i$: each edge of P_i reached from d_i in R_i
- Design a route servicing arcs of $\overleftrightarrow{P_i}$ by a single vehicle: start, end in d_i, only $\overleftrightarrow{R_i}$ used.
- Several requirements: (1) max length of a route, (2) road priorities: how many times a road serviced, (3) capacity: after some length back to d_i (4) minimize complaints

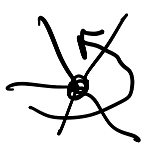
Practically: All two-ways roads, $P_i = R_i \ni d_i$ subtrees of G

? Does a route exist? is hard

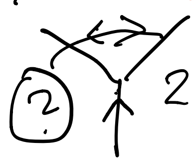
Complaints: $\boxed{\text{rational ass.}}$ that complaints can be deduced from the vehicle route and its perceived unfairness

Frustration of a vehicle route

Added decoration : cyclic order around each vertex

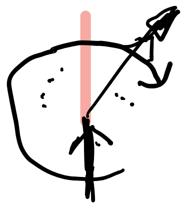


If no consensus then ~~more cars required by admin~~

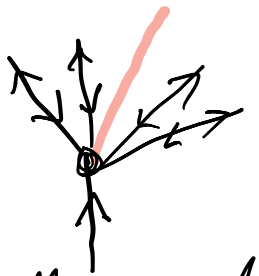


2 cars

forward complaint



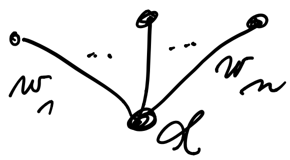
backward complaint



both complaints

Frustration of a route # complaints

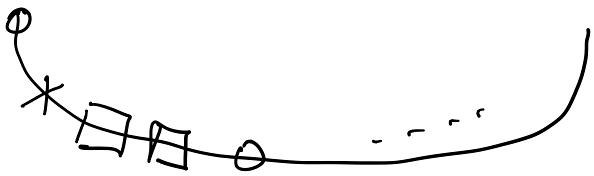
$\boxed{\text{Find admissible route minimising frustration}}$

- Necklace splitting reducible to 

- w_i natural (length) also non-polynomial reflect hard mountain conditions
- **Natural Question** : necklace splitting for graphs different from path.

Discussions with Martin Tancer (lecture for REU 21, see web page of CoSP.

Necklace Splitting



- $k \cdot n$ vertices-beads of s different colors
- $k \cdot a_i$ of color i

- k-splitting: partition into k parts, each consisting of a finite number of disjoint intervals, each part has exactly a_i beads of color i, $i = 1, \dots, s$.

Frustration: min # cuts needed

Theorem (Alon) There is always splitting of frustration $\leq (k-1) \cdot s$.

Non-constructive, topological methods

! Accompany fairness !

Algorithmic Complexity

- smallest # cuts NP_c even for $k=2$, $a_i = 1$
- Find Alon's solution: extensive research,

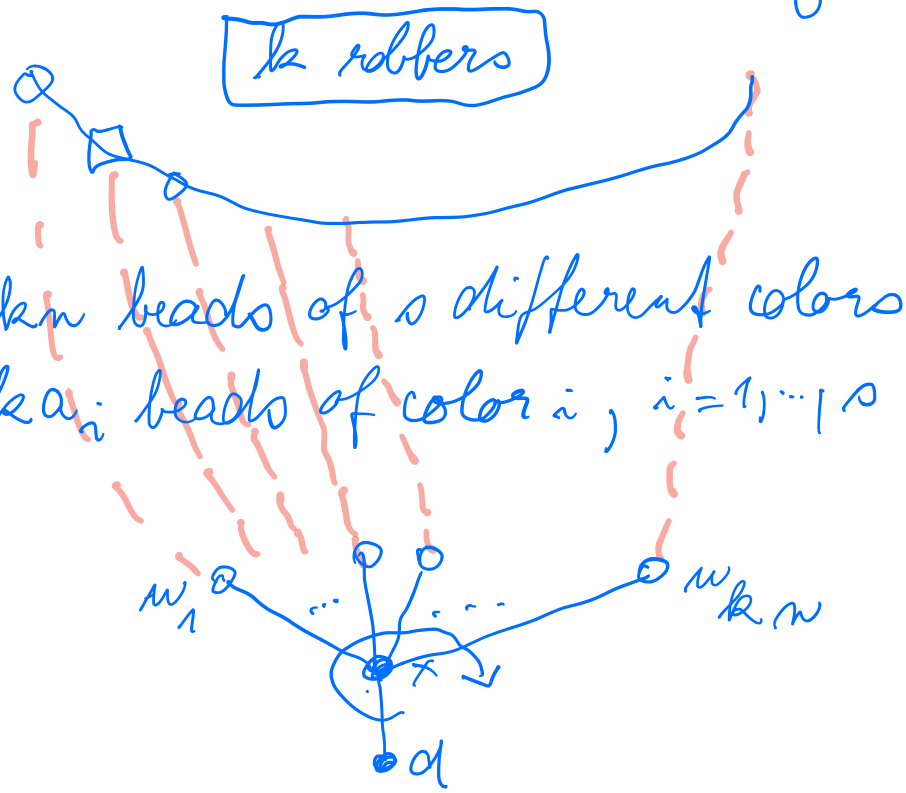
Theorem (Filos-Ratsikas, Goldberg 2019)

PPA-complete even for $k=2$.

[cryptographic hardness]

Approximations studied

Frustration of a route and necklace splitting



kn beads of s different colors

ka_i beads of color i, $i = 1, \dots, s$

- define weights $M_1, \dots, M_s$

$$M_1 := 1 + n$$
$$M_{r+1} := 1 + n \sum_{q \le r} M_q \qquad \| \quad \bullet\ w(x,d) = 1$$

- let $w(x w_i) = M_i$ if i-th bead has color i.

Want:

- Route length $\le L = 2k\left(1 + \sum a_r M_r\right)$
- $c = 1/k$

$\Rightarrow$ the length of each trip to depot must be exactly $2\left(1 + \sum_{r=1}^{s} a_r M_r\right)$. $\square$

Open Problem (Loebl, Tancer)

Necklace, t types of beads, k robbers

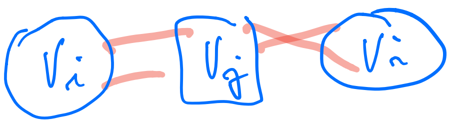
|||

graph $G=(V,E)$, $c: V(G) \to \{1,\dots,t\}$

$|\{v;\ c(v)=i\}|$ is divisible by k.

Split $V(G)$ into $V_1,\dots,V_k$ so that for each color i there is the same number of beads of color i in each V_j.

Want minimise # edges which are not a subset of some V_i.



Non-trivial even for trees

Fair division of indivisible items

X finite set of items, Agents $1, \dots, k$

$u_i : 2^X \longrightarrow \mathbb{Q}$ utility fction of agent i

- monotone, submodular (STTI Vondrák's lecture)

- Allocation: partition of X to bundles of agents
- Envy-free (EF)
  $u_i(i\text{'s bundle}) \geq u_i(j\text{'s bundle})$
- Proportional (P)
  $u_i(i\text{'s bundle}) \geq u_i(X)$
- Envy-free up to one good (EF1)
  For each i, j there is good $o(i,j)$ in j's bundle
  $u_i(i\text{'s bundle}) \geq u_i(j\text{'s bundle} \setminus \{o(i,j)\})$

Fair Division of a graph

$G = (V, E)$ graph; $X = V$; $u_i : V \longrightarrow \mathbb{Q}^+$, $i = 1, \dots, k$

$U \subseteq V \Rightarrow u_i(U) = \sum u_i(v)$; $v \in U$ if $G[U]$ connected

Theorem (Bilò et al 2019) G a path

- EF1 allocation exists if $k \leq 4$.
  $k = 4 \Rightarrow$ Sperner's lemma is used.
- EF2 allocations exist for general k

- The deleted vertex on the boundary

Open Problem

- For the case of three or more agents and non-Hamiltonian graphs, characterise the class of graphs guaranteeing EF_i, $i \geq 1$.

- Compare topological methods of Bilò et al with the necklace splitting methods.

A distribution system for crises

- Motivated by chaotic distribution of critical medical supplies world-wide {KAM, ethics law}
- Grant from MV (interior ministry)

1 Another fairness set-up:

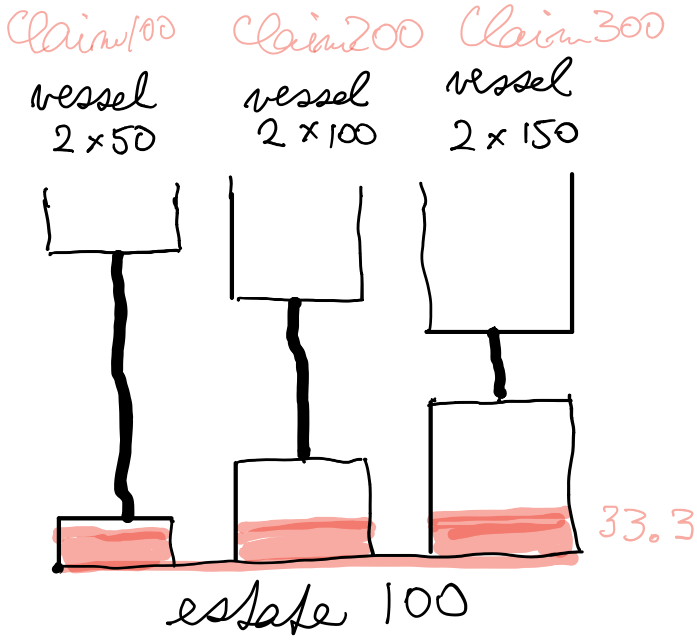
- X amount of good [divisible or indivisible]
- Agent i demands d_i crisis: $\sum d_i \gg |X|$
- Solution: function (algorithm)

$f^X : \{1, \dots, k\} \to \mathbb{R}$; $f(i)$ = how much of X i gets

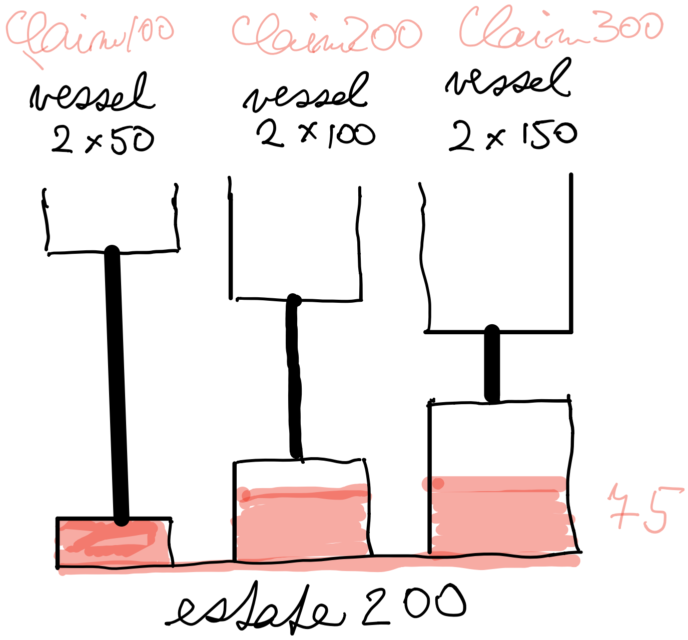
Hence $\boxed{\sum f(i) = X}$

Fairness of a solution:

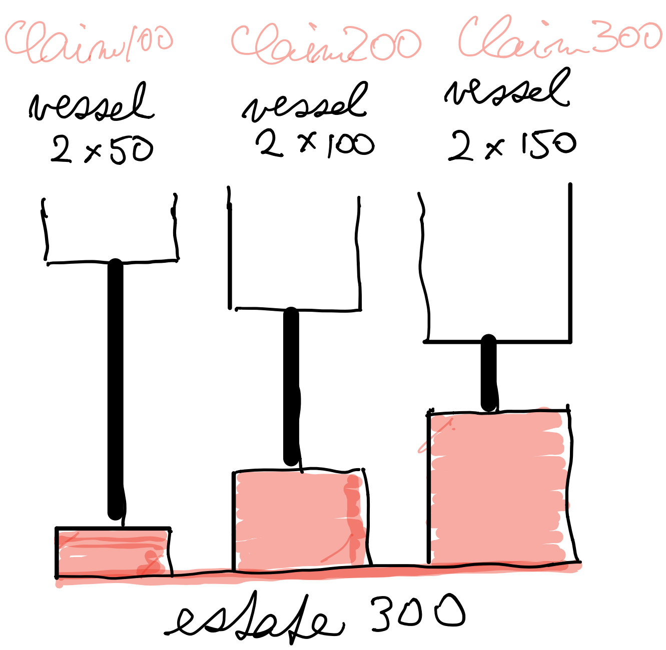
- Monotone $|X| \le |X'| \Rightarrow f^X(i) \le f^{X'}(i)$
- Consistent for all pairs i, j, if they divide $X(i,j) = f^X(i) + f^X(j)$ by the same algorithm, the output is $f^X(i)$, $f^X(j)$.
- Self-dual losses distributed in the same way as gains
- Other properties make the solution unique



Pouring liquid of 100
to the connected
vessels of claims.



Pouring liquid of 200
to the connected
vessels of claims.



Pouring liquid of 300
to the connected
vessels of claims.

Median of Partitions

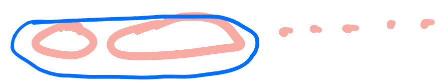
P_1

P_k

Find a partition M (median) with min total distance to $P_1, \dots, P_k$:

$$\min \sum_{i=1}^{k} \mathrm{dist}(M, P_i)$$

dist(P, Q): min # operations to get P from Q.

1. 
2.