

COOPERATIVE GAME THEORY

MARTIN ČERNÝ

KAM.MFF.CUNI.CZ/~CERNY
CERNY@KAM.MFF.CUNI.CZ

MARCH 20, 2023

GENERALISED MODELS OF COOPERATION

Cooperative game

A **cooperative game** is an ordered pair (N, v) , where N is a set of players and $v: 2^N \rightarrow \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

Disadvantages of the model:

1. **we might not be interested in forming N**
 - ▶ we want to analyse coalition formation
2. **not every coalition makes sense**
 - ▶ $\{H, I\}$... hardware + IT support
 - ▶ why should we consider $x(\{H, I\}) \geq v(\{H, I\})$?
3. **2^n real values representing v**
 - 3.1 unreal (expensive) to get all information
 - 3.2 expensive to store all information
4. **not suitable for all scenario**
 - ▶ unsuitable for more complex scenarios

Cooperative game

A **cooperative game** is an ordered pair (N, v) , where N is a set of players and $v: 2^N \rightarrow \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

Disadvantages of the model:

1. **we might not be interested in forming N**
 - ▶ Games with coalition structure
2. **not every coalition makes sense**
 - ▶ Restricted games
3. **2^n real values representing v**
 - ▶ Uncertainty models
 - ▶ Models with *compact* characteristic function
4. **not suitable for all scenario**
 - ▶ Bi-cooperative games and games with overlapping coalitions

1. NOT INTERESTED IN FORMING N

- Goal: Understand cooperation outside N

Cooperative games with coalition structure

Cooperative game with coalition structure (N, v, \mathcal{B}) is tuple, where N is the player set, $v: 2^N \rightarrow \mathbb{R}$ is the *characteristic function* and \mathcal{B} is a partition of N . It holds $v(\emptyset) = 0$.

- $\mathcal{B} = \{B_1, \dots, B_k\}$
 - ▶ B_i ... coalition willing to cooperate
- How to distribute payoff here?

- Goal: Understand cooperation outside N

Cooperative games with coalition structure

Cooperative game with coalition structure (N, v, \mathcal{B}) is tuple, where N is the player set, $v: 2^N \rightarrow \mathbb{R}$ is the *characteristic function* and \mathcal{B} is a partition of N . It holds $v(\emptyset) = 0$.

- How to distribute payoff here?
- $\mathcal{I}_{\mathcal{B}} = \{x \in \mathbb{R}^n \mid x(B_i) = v(B_i) \text{ for } B_i \in \mathcal{B} \text{ and } x_i \geq v(i) \text{ for } i \in N\}$

GAMES WITH COALITION STRUCTURE - FAIR SOLUTION

- If $f(v): \Gamma_{\mathcal{B}}^n \rightarrow \mathbb{R}^n$ is a single-point solution satisfying
 1. (AXIOM OF **RELATIVE EFFICIENCY**)
 - ▶ $\forall B_j \in \mathcal{B} : \sum_{i \in B_j} f_i(v) = v(B_j)$
 - ▶ *We split the value of coalition between its players.*
 2. (AXIOM OF **SYMMETRY**)
 - ▶ $\forall \sigma \in \Sigma_n (\forall i, j : i \in B_j \implies \sigma(i) \in B_j) \implies f(v_\sigma)(S) = f(v)(\sigma(S))$
 - $v_\sigma(S) := v(\sigma(S))$
 - $\sigma(S) := \{\sigma(\ell) \mid \ell \in S\}$
 - ▶ *Same value, same payment.*
 3. (AXIOM OF **NULL PLAYER**)
 - ▶ $\forall i \in N : (\forall S \subseteq N : v(S) = v(S \cup i)) \implies f_i(v) = 0$
 - ▶ *No pain, no gain.*
 4. (AXIOM OF **ADDITIVITY**)
 - ▶ $\forall v, w \in \Gamma^n : f(v + w) = f(v) + f(w)$
 - ▶ *Division to simple games lead to same payments.*
- then it is **uniquely determined!**

GAMES WITH COALITION STRUCTURE

Cooperative games with coalition structure

Cooperative game with coalition structure (N, v, \mathcal{B}) is tuple, where N is the player set, $v: 2^N \rightarrow \mathbb{R}$ is the *characteristic function* and \mathcal{B} is a partition of N . It holds $v(\emptyset) = 0$.

- How to distribute payoff here?
- $\mathcal{I}_{\mathcal{B}} = \{x \in \mathbb{R}^n \mid x(B_i) = v(B_i) \text{ for } B_i \in \mathcal{B} \text{ and } x_i \geq v(i) \text{ for } i \in N\}$

The Shapley value

For a cooperative game with coalition structure (N, v, \mathcal{B}) , the Shapley value $\phi_{\mathcal{B}}: \Gamma_{\mathcal{B}}^n \rightarrow \mathbb{R}^n$ is defined for $i \in B_j$ as

$$\phi_{\mathcal{B}}(v)_i := \phi(v_{B_j})_i.$$

- Equivalently, we compute $\phi(v_{B_j})$ for $B_j \in \mathcal{B}$
- Similar with most of the standard solution concepts

2. NOT EVERY COALITION MAKES SENSE

- Goal: Consider some coalitions are *infeasible*

Restricted cooperative games

Restricted cooperative games (N, v, \mathcal{F}) is tuple, where N is a player set, $v: 2^N \rightarrow \mathbb{R}$ is the *characteristic function* and $\mathcal{F} \subseteq 2^N$ is the set of feasible coalitions. It holds $\emptyset \in \mathcal{F}$ a $v(\emptyset) = 0$.

- How to distribute the payoff here?

RESTRICTED GAMES

- Goal: Consider some coalitions are *infeasible*

Restricted cooperative games

Restricted cooperative games (N, v, \mathcal{F}) is tuple, where N is a player set, $v: 2^N \rightarrow \mathbb{R}$ is the *characteristic function* and $\mathcal{F} \subseteq 2^N$ is the set of feasible coalitions. It holds $\emptyset \in \mathcal{F}$ and $v(\emptyset) = 0$.

Core of restricted games

For a restricted cooperative game (N, v, \mathcal{F}) , the core $\mathcal{C}_{\mathcal{F}}$ is defined as

$$\mathcal{C}_{\mathcal{F}}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S) \text{ pro } S \in \mathcal{F}\}.$$

- No need to consider $x(S) \geq v(S)$ for $S \notin \mathcal{F}$

3.1 UNREAL (EXPENSIVE) TO GET ALL INFORMATION

- Goal: To model situation where it is not possible to determine $v(S)$
 1. exactly
 2. cheaply
 3. at all

Possible solutions:

- ▶ interval games
- ▶ incomplete games
- ▶ stochastic games
 - more next week

COOPERATIVE INTERVAL GAMES

- Goal: To model situation where it is not possible to determine $v(S)$ exactly
 - ▶ \mathbb{IR} ... set of real closed vectors
 - ▶ $\mathbf{x} \in \mathbb{IR} : \mathbf{x} = [\underline{x}, \bar{x}]$

Cooperative interval game

Cooperative interval game (N, w) is a pair, where N is the player set and $w: 2^N \rightarrow \mathbb{IR}$ is the characteristic function. It holds $w(\emptyset) = [0, 0]$.

- Two approaches:
 1. Weak ordering of intervals
 - $x, y \in \mathbb{IR} : x \succeq y \iff \underline{x} \geq \underline{y} \text{ a } \bar{x} \geq \bar{y}$
 2. Selections
 - (N, v) is a **selection** (N, w) , if $v(S) \in w(S)$ for $\forall S \subseteq N$

- *Rather theoretical...*

Interval imputation

Set of **interval imputations** $\mathcal{I}(w)$ of a cooperative interval game (N, w) is defined as

$$\mathcal{I}(w) := \{(\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n) \in \mathbb{I}\mathbb{R}^n \mid \sum_{i \in N} \mathbf{l}_i = w(N), \mathbf{l}_i \succeq w(i), \forall i \in N\}.$$

Interval core

The **interval core** $\mathcal{C}(w)$ of a cooperative interval game (N, w) is defined as

$$\mathcal{C}(w) := \{(\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n) \in \mathcal{I}(w) \mid \sum_{i \in S} \mathbf{l}_i \succeq w(S), \text{ pro } S \subseteq N\}.$$

- *Enumerating all possibilities*

Interval selection imputation

The set of **interval selection imputations** $\mathcal{SI}(w)$ of a cooperative interval game (N, w) is defined as

$$\mathcal{SI}(w) := \bigcup \{ \mathcal{I}(v) \mid (N, v) \text{ is selection } (N, w) \}.$$

Interval selection core

The **interval selection core** $\mathcal{SC}(w)$ of a cooperative interval game (N, w) is defined as

$$\mathcal{SC}(w) := \bigcup \{ \mathcal{C}(v) \mid (N, v) \text{ is selection } (N, w) \}.$$

COOPERATIVE INTERVAL GAMES - RELATIONS AND QUESTIONS

■ The difference between $\mathcal{C}(w)$ and $\mathcal{SC}(w)$:

- ▶ $\mathbf{x} \in \mathcal{C}(w) \implies \mathbf{x} \in \mathbb{IR}^n$
- ▶ $x \in \mathcal{SC}(w) \implies x \in \mathbb{R}^n$

■ Questions:

1. description of $\mathcal{C}(w)$ and $\mathcal{SC}(w)$
2. when $\mathcal{C}(w) = \mathcal{SC}(w)$?
 - 2.1 Does $\mathcal{SC}(w) \subseteq \mathcal{C}(w)$ hold for every game?
 - 2.2 ...

INCOMPLETE COOPERATIVE GAME

- Goal: To model situations where it is not possible to determine $v(S)$ at all
 - ▶ alternatively, it is too costly (time, money)

Incomplete cooperative game

The incomplete cooperative game (N, \mathcal{K}, v) is a tuple, where N is the player set, $\mathcal{K} \subseteq 2^N$ is the set of coalitions with known value and $v: \mathcal{K} \rightarrow \mathbb{R}$ is the characteristic function. It holds $\emptyset \in \mathcal{K}$ a $v(\emptyset) = 0$.

- Restricted game (N, \mathcal{F}, v) :
 - ▶ the same model
 - ▶ different interpretation of \mathcal{F} and \mathcal{K}
 - \mathcal{F} ... feasible coalitions, other coalitions are **unfeasible** to form
 - \mathcal{K} ... known coalitions, values of other coalitions are **unknown**

INCOMPLETE COOPERATIVE GAME

- Goal: To model situations where it is not possible to determine $v(S)$ at all
 - ▶ alternatively, it is too costly (time, money)

Incomplete cooperative game

The incomplete cooperative game (N, \mathcal{K}, v) is a tuple, where N is the player set, $\mathcal{K} \subseteq 2^N$ is the set of coalitions with known value and $v: \mathcal{K} \rightarrow \mathbb{R}$ is the characteristic function. It holds $\emptyset \in \mathcal{K}$ a $v(\emptyset) = 0$.

- Questions:
 1. *What does the underlying complete game look like, based on knowing something about its properties?*
 - \implies C-extension
 2. *How to split the payoff between players based on incomplete information?*
 - weak and strong solution concepts

INCOMPLETE COOPERATIVE GAME

- Goal: To model situations where it is not possible to determine $v(S)$ at all
 - ▶ alternatively, it is too costly (time, money)

Incomplete cooperative game

The incomplete cooperative game (N, \mathcal{K}, v) is a tuple, where N is the player set, $\mathcal{K} \subseteq 2^N$ is the set of coalitions with known value and $v: \mathcal{K} \rightarrow \mathbb{R}$ is the characteristic function. It holds $\emptyset \in \mathcal{K}$ a $v(\emptyset) = 0$.

- $C \subseteq \Gamma^n$... class of cooperative games

C-extension

Cooperative game (N, w) is C-extension, if $(N, w) \in C$ and $w(S) = v(S)$ for every $S \in \mathcal{K}$.

- Goal: To analyse the set of C-extensions.

INCOMPLETE COOPERATIVE GAME - THE WEAK AND THE STRONG SOLUTION

- $B^n(v)$... set of **balanced** extensions (N, \mathcal{K}, v)

We do not want to leave anything out...

Weak core of incomplete game

The weak core $\bigcup \mathcal{C}$ of an incomplete game (N, \mathcal{K}, v) is defined as

$$\bigcup \mathcal{C}(v) := \bigcup_{(N,w) \in B^n(v)} \mathcal{C}(w).$$

We want to be 100% sure we select something feasible

Strong core of incomplete game

The strong core $\bigcap \mathcal{C}$ of an incomplete game (N, \mathcal{K}, v) is defined as

$$\bigcap \mathcal{C}(v) := \bigcap_{(N,w) \in B^n(v)} \mathcal{C}(w).$$

- Goal: To model situations where it is not possible to determine $v(S)$ exactly
 - ▶ ζ ... random variable
 - ▶ Y_ζ ... range of ζ

Stochastic cooperative game

Stochastic cooperative game (N, v) is pair, where N is the player set and $v: 2^N \times Y_\zeta \rightarrow \mathbb{R}$ is the characteristic function.

- More by David Ryzák next time

- Goal: Compact representation $v: 2^N \rightarrow \mathbb{R}$
 - ▶ $G = (N, E, w)$... weighted graph
 - N ... vertices
 - E ... edges
 - $w: E \rightarrow \mathbb{R}_+$... weight function

Graph cooperative game

Graph cooperative game (G, v) is a pair, where $G = (N, E, w)$ is a weighted graph and $v: 2^N \rightarrow \mathbb{R}$ is the characteristic function defined as

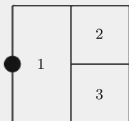
$$v(S) := \sum_{i,j \in S: \{i,j\} \in E} w(\{i,j\}).$$

4. OTHER GENERALISATIONS OF COOP. GAME MODELS

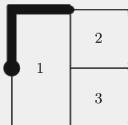
BI-COOPERATIVE GAMES - MOTIVATION

■ Situation: Farmers want to build irrigation system

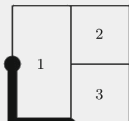
Case a (cost=1)



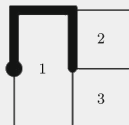
Case b (cost=11)



Case c (cost=11)



Case d (cost=16)



- Goal: To capture positive and negative *contributions* to games

- ▶ $Q(N) = \{(S, T) \mid S, T \subseteq N, S \cap T = \emptyset\}$

Bi-cooperative game

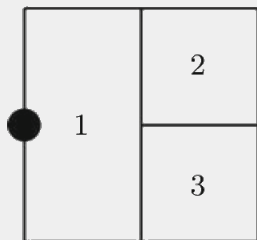
A bi-cooperative game (N, v) is pair, where N is the player set and $v: Q(N) \rightarrow \mathbb{R}$ is the characteristic function. It holds $v(\emptyset, \emptyset) = 0$.

BI-COOPERATIVE GAMES

Example:

- $v(S, T)$... price of the irrigation system if
 - ▶ S ... want water for their land
 - ▶ T ... want water for their land and allow the piping to be build on their land

Case a (cost=1)



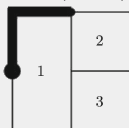
- - ▶ $v(\{1\}, \emptyset) = v(\emptyset, \{1\}) = 1$

BI-COOPERATIVE GAMES

Example:

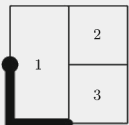
- $v(S, T)$... price of the irrigation system if
 - ▶ S ... want water for their land
 - ▶ T ... want water for their land and allow the piping to be build on their land

Case b (cost=11)



- ▶ $v(\{1, 2\}, \emptyset) = v(\{1\}, \{2\}) = v(\{2\}, \emptyset) = 11$

Case c (cost=11)



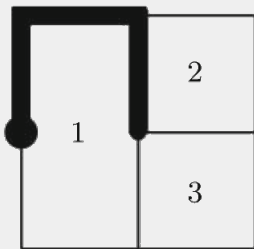
- ▶ $v(\{1, 3\}, \emptyset) = v(\{1\}, \{3\}) = v(\{3\}, \emptyset) = 11$

BI-COOPERATIVE GAMES - SHAPLEY VALUE

Example:

- $v(S, T)$... price of the irrigation system if
 - ▶ S ... want water for their land
 - ▶ T ... want water for their land and allow the piping to be build on their land

Case d (cost=16)



- - ▶ $v(\{1, 2, 3\}, \emptyset) = v(\{1\}, \{2, 3\}) = v(\{2, 3\}, \emptyset) = 16$

BI-COOPERATIVE GAMES - SHAPLEY VALUE

Bi-cooperative game

A bi-cooperative game (N, v) is pair, where N is the player set and $v: \mathcal{Q}(N) \rightarrow \mathbb{R}$ is the characteristic function. It holds $v(\emptyset, \emptyset) = 0$.

Shapley value

The Shapley value $\phi^{\mathcal{Q}}$ of a bi-cooperative game (N, v) is defined as

$$\phi_i^{\mathcal{Q}} := \phi_i^+ + \phi_i^-,$$

where

- $\phi_i^+ := \sum_{(S,T) \subseteq \mathcal{Q}(N \setminus i)} \alpha_{(S,T)} [v(S \cup i, T) - v(S, T)],$
- $\phi_i^- := \sum_{(S,T) \subseteq \mathcal{Q}(N \setminus i)} \beta_{(S,T)} [v(S, T \cup i) - v(S, T)].$

- Goal: Distribute *recourses* of players between several coalitions
 - ▶ time, money, materials, ...

Game with overlapping coalitions

Cooperative game with overlapping coalitions (N, v) is pair, where N is the player set and $v: [0, 1]^n \rightarrow \mathbb{R}$ is the characteristic function. It holds $v(\mathbf{0}^n) = 0$.

- Definitions of solution concepts are not standardised
- my opinion: cumbersome model

Summary

There is a variety of models generalising the standard model of cooperative games. There are models concerned with coalition structures, uncertainty in data, compact representations, or modelling more complex situations. The list of models we saw today is not exhaustive.