Cooperative game theory

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Generalised models of cooperation

Cooperative game

A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

Disadvantages of the model:

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   - why should we consider $x(\{H, I\}) \geq v(\{H, I\})$?

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   - unsuitable for more complex scenarios

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   - Bi-cooperative games and games with overlapping coalitions

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Games with coalition structure

- Goal: Understand cooperation outside N

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GAMES WITH COALITION STRUCTURE - FAIR SOLUTION

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- then it is **uniquely determined!**

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The Shapley value

For a cooperative game with coalition structure $(N, v, \mathcal{B})$, the Shapley value $\phi_{\mathcal{B}}\colon \Gamma^n_{\mathcal{B}} \to \mathbb{R}^n$ is defined for $i \in \mathcal{B}_j$ as

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- Similar with most of the standard solution concepts

2. Not every coalition makes sense

- Goal: Consider some coalitions are *infeasible*

Restricted cooperative games

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Core of restricted games

For a restricted cooperative game $(N, v, \mathcal{F})$, the core $\mathcal{C}_{\mathcal{F}}$ is defined as

$$\mathcal{C}_{\mathcal{F}}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S) \text{ pro } S \in \mathcal{F}\}.$$

- No need to consider $x(S) \geq v(S)$ for $S \notin \mathcal{F}$

3.1 Unreal (expensive) to get all information

Models dealing with uncertainty

- Goal: To model situation where it is not possible to determine $v(S)$

  Possible solutions:

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    - more next week

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  - $\mathbb{IR}$ ... set of real closed vectors
  - $\mathbf{x} \in \mathbb{IR} : \mathbf{x} = [\underline{x}, \overline{x}]$

Cooperative interval game

Cooperative interval game (N, w) is a pair, where N is the player set and $w\colon 2^N \to \mathbb{IR}$ is the characteristic function. It holds $w(\emptyset) = [0, 0]$.

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  2. Selections
     - (N, v) is a **selection** (N, w), if $v(S) \in w(S)$ for $\forall S \subseteq N$

- *Rather theoretical...*

Interval imputation

Set of **interval imputations** $\mathcal{I}(w)$ of a cooperative interval game (N, w) is defined as

$$\mathcal{I}(w) := \{(\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n) \in \mathbb{IR}^n \mid \sum_{i \in N} \mathbf{I}_i = w(N), \mathbf{I}_i \succeq w(i), \forall i \in N\}.$$

Interval core

The **interval core** $\mathcal{C}(w)$ of a cooperative interval game (N, w) is defined as

$$\mathcal{C}(w) := \{(\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n) \in \mathcal{I}(w) \mid \sum_{i \in S} \mathbf{I}_i \succeq w(S), \text{pro } S \subseteq N\}.$$

- *Enumerating all possibilites*

Interval selection inputation

The set of **interval selection imputations** $\mathcal{SI}(w)$ of a cooperative interval game (N, w) is defined as

$$\mathcal{SI}(w) := \bigcup \{\mathcal{I}(v) \mid (N, v) \text{ is selection } (N, w)\}.$$

Interval selection core

The **interval selection core** $\mathcal{SC}(w)$ of a cooperative interval game (N, w) is defined as

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Cooperative interval games - relations and questions

- The difference between $\mathcal{C}(w)$ and $\mathcal{SC}(w)$:

- Questions:

Cooperative interval games - relations and questions

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  - $\mathbf{x} \in \mathcal{C}(w) \implies \mathbf{x} \in \mathbb{IR}^n$
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Cooperative interval games - relations and questions

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  - $\mathbf{x} \in \mathcal{C}(w) \implies \mathbf{x} \in \mathbb{IR}^n$
  - $x \in \mathcal{SC}(w) \implies x \in \mathbb{R}^n$
- Questions:
  1. description of $\mathcal{C}(w)$ and $\mathcal{SC}(w)$
  2. when $\mathcal{C}(w) = \mathcal{SC}(w)$?
     - 2.1 Does $\mathcal{SC}(w) \subseteq \mathcal{C}(w)$ hold for every game?
     - 2.2 . . .

- Goal: To model situations where it is not possible to determine $v(S)$ at all
  - alternatively, it is too costly (time, money)

Incomplete cooperative game

The incomplete cooperative game $(N, \mathcal{K}, v)$ is a tuple, where N is the player set, $\mathcal{K} \subseteq 2^N$ is the set of coalitions with known value and $v\colon \mathcal{K} \to \mathbb{R}$ is the characteristic function. It holds $\emptyset \in \mathcal{K}$ a $v(\emptyset) = 0$.

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  - the same model
  - different interpretation of $\mathcal{F}$ and $\mathcal{K}$

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- Restricted game $(N, \mathcal{F}, v)$:
  - the same model
  - different interpretation of $\mathcal{F}$ and $\mathcal{K}$
    - $\mathcal{F}$ ... feasible coalitions, other coalitions are **unfeasible** to form
    - $\mathcal{K}$ ... known coalitions, values of other coalitions are **unknown**

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- Questions:
  1. *What does the underlying complete game look like, based on knowing something about its properties?*
  2. *How to split the payoff between players based on incomplete information?*

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- Questions:
  1. *What does the underlying complete game look like, based on knowing something about its properties?*
     - $\implies$ C-extension
  2. *How to split the payoff between players based on incomplete information?*
     - weak and strong solution concepts

Incomplete cooperative game

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The incomplete cooperative game $(N, \mathcal{K}, v)$ is a tuple, where N is the player set, $\mathcal{K} \subseteq 2^N$ is the set of coalitions with known value and $v\colon \mathcal{K} \to \mathbb{R}$ is the characteristic function. It holds $\emptyset \in \mathcal{K}$ a $v(\emptyset) = 0$.

- $C \subseteq \Gamma^n$ ... class of cooperative games

C-extension

Cooperative game (N, w) is C-extension, if $(N, w) \in C$ and

$$w(S) = v(S) \text{ for every } S \in \mathcal{K}.$$

- Goal: To analyse the set of C-extensions.

Incomplete cooperative game - The weak and the strong solution

- $B^n(v)$ ... set of **balanced** extensions $(N, \mathcal{K}, v)$

We do not want to leave anything out...

Weak core of incomplete game

The weak core $\bigcup \mathcal{C}$ of an incomplete game $(N, \mathcal{K}, v)$ is defined as

$$\bigcup \mathcal{C}(v) := \bigcup_{(N,w) \in B^n(v)} \mathcal{C}(w).$$

We want to be 100% sure we select something feasible

Strong core of incomplete game

The strong core $\bigcap \mathcal{C}$ of an incomplete game $(N, \mathcal{K}, v)$ is defined as

$$\bigcap \mathcal{C}(v) := \bigcap_{(N,w) \in B^n(v)} \mathcal{C}(w).$$

- Goal: To model situations where it is not possible to determine $v(S)$ exactly
  - ζ ... random variable
  - Y_ζ ... range of ζ

Stochastic cooperative game

Stochastic cooperative game (N, v) is pair, where N is the player set and $v\colon 2^N \times Y_\zeta \to \mathbb{R}$ is the characteristic function.

- More by David Ryzák next time

- Goal: Compact representation $v\colon 2^N \to \mathbb{R}$
  - $G = (N, E, w)$ ... weighted graph
    - N ... vertices
    - E ... edges
    - $w\colon E \to \mathbb{R}_+$ ... weight function

Graph cooperative game

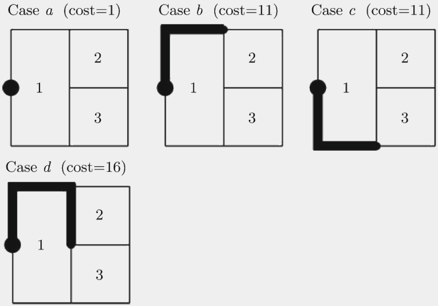
Graph cooperative game (G, v) is a pair, where $G = (N, E, w)$ is a weighted graph and $v\colon 2^N \to \mathbb{R}$ is the characteristic function defined as

$$v(S) := \sum_{i,j \in S : \{i,j\} \in E} w(\{i, j\}).$$

4. Other generalisations of coop. game models

Bi-cooperative games - motivation

- Situation: Farmers want to build irrigation system



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- Goal: To capture positive and negative *contributions* to games
  - $\mathcal{Q}(N) = \{(S, T) \mid S, T \subseteq N, S \cap T = \emptyset\}$

Bi-cooperative game

A bi-cooperative game (N, v) is pair, where N is the player setand $v\colon \mathcal{Q}(N) \to \mathbb{R}$ is the characteristic function. It holds$v(\emptyset, \emptyset) = 0$.

Bi-cooperative games

Example:

- $v(S, T)$ ... price of the irrigation system if
  - S ... want water for their land
  - T ... want water for their land and allow the piping to be build on their land

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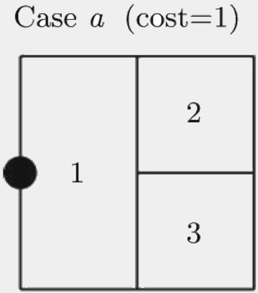
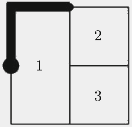


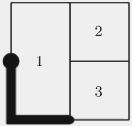
-
  - $v(\{1\}, \emptyset) = v(\emptyset, \{1\}) = 1$

Example:

- $v(S,T)$ ... price of the irrigation system if
  - S ... want water for their land
  - T ... want water for their land and allow the piping to be build on their land
- Case b (cost=11)

  - $v(\{1,2\},\emptyset) = v(\{1\},\{2\}) = v(\{2\},\emptyset) = 11$
- Case c (cost=11)

  - $v(\{1,3\},\emptyset) = v(\{1\},\{3\}) = v(\{3\},\emptyset) = 11$

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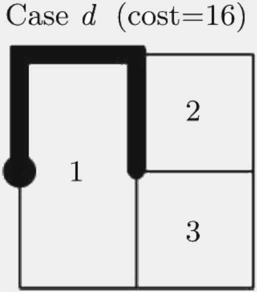

-
  - $v(\{1,2,3\},\emptyset) = v(\{1\},\{2,3\}) = v(\{2,3\},\emptyset) = 16$

Bi-cooperative game

A bi-cooperative game (N, v) is pair, where N is the player setand $v\colon \mathcal{Q}(N) \to \mathbb{R}$ is the characteristic function. It holds$v(\emptyset, \emptyset) = 0$.

Shapley value

The Shapley value $\phi^{\mathcal{Q}}$ of a bi-cooperative game (N, v) is defined as

$$\phi_i^{\mathcal{Q}} := \phi_i^+ + \phi_i^-,$$

where

- $\phi_i^+ := \sum_{(S,T)\subseteq\mathcal{Q}(N\setminus i)} \alpha_{(S,T)} [v(S \cup i, T) - v(S, T)]$,
- $\phi_i^- := \sum_{(S,T)\subseteq\mathcal{Q}(N\setminus i)} \beta_{(S,T)} [v(S, T \cup i) - v(S, T)]$.

- Goal: Distribute *recources* of players between several coalitions
  - time, money, materials, ...

Game with overlapping coalitions

Cooperative game with overlapping coalitions (N, v) is pair, where N is the player set and $v\colon [0,1]^n \to \mathbb{R}$ is the characteristic function. It holds $v(0^n) = 0$.

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  - time, money, materials, ...

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- Definitions of solution concepts are not standardised
- my opinion: cumbersome model

Summary

There is a variety of models generalising the standard model of cooperative games. There are models concerned with coalition structures, uncertainty in data, compact represetations, or modelling more complex situations. The list of models we saw today is not exhaustive.