Cooperative game theory

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Classes of games

Motivation

1. $\mathcal{C}(v) \neq \emptyset \iff (N, v)$ is **balanced**
   - *But what are* ***balanced*** *games?*
2. $\eta(v) \neq \emptyset \iff (N, v)$ is **essential**
   - *But what are* ***essential*** *games?*
3. $\eta(v) \subseteq \mathcal{C}(v) \subseteq \mathcal{K}(v) \subseteq (v)$
   - *But when does* $=$ *or* $\subsetneq$ *hold?*

A cooperative game (N, v) is

1. **monotone** if $v(S) \subseteq v(T)$ for $S \subseteq T \subseteq N$,
2. **weakly superadditive** if $v(S) + v(i) \leq v(S \cup i)$ for $S \subseteq N \setminus \{i\}$,
3. **superadditive** if $v(S) + v(T) \leq v(S \cup T)$ for $S, T \subseteq N$, $S \cap T = \emptyset$,
4. **convex** if $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$, for $S, T \subseteq N$,
5. **positive** if $m^v(S) \geq 0$ for every $S \subseteq N$,
    - $m^v(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T)$,
6. **essential** if $v(N) \geq \sum_{i \in N} v(i)$,
7. **balanced** if $\mathcal{C}(v) \neq \emptyset$.

What is the relation between the classes?

Relations between classes I.

Balanced cooperative games are **essential**.

- (N, v) is **essential**
  - $\mathcal{I}(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x_i \geq v(i), i \in N\} \neq \emptyset$
- (N, v) is **balanced**
  - $\emptyset \neq \mathcal{C}(v) \subseteq \mathcal{I}(v)$

Relations between classes II.

Positive cooperative games are **convex**.

- $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$
- $\sum_{X \subseteq S} m^v(X) + \sum_{X \subseteq T} m^v(X) \leq \sum_{X \subseteq S \cap T} m^v(X) + \sum_{X \subseteq S \cup T} m^v(X)$
- $0 \leq \sum_{X \subseteq S \cup T, X \not\subseteq S, X \not\subseteq T} m^v(X)$
  - holds since $m^v(X) \geq 0$ for every $X \subseteq N$

Relations between classes III.

Convex cooperative games are **superadditive**.

- $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$
- $S \cap T = \emptyset$
  - $v(S \cap T) = 0$
  - $v(S) + v(T) \leq v(S \cup T)$

Relations between classes IV.

Superadditive cooperative games are **weakly superadditive**.

- $v(S) + v(T) \leq v(S \cup T)$, $S \cap T = \emptyset$
- $T = \{i\}$
  - $v(S) + v(i) \leq v(S \cup i)$

Example: $N = \{1, 2\}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$v(S)$	0	-1	1	0

1. $v(1) + v(2) \leq v(\{1,2\})$
   - $\implies$ **weakly superadditive**
2. $v(\emptyset) \not\leq v(\{1\})$
   - $\implies$ **not monotone**

- **Negative values** are the problem!

Classes of games V.

Non-negative **weakly superadditive** games are **monotone**.

- $T \subseteq S \subseteq N \qquad S \setminus T = \{i_1, \dots i_k\}$
  1. $v(T) + v(i_1) \leq v(T \cup i_1)$
  2. $v(T \cup i_1) + v(i_2) \leq v(T \cup \{i_1, i_2\})$
  3. $\vdots$
  4. $v(S \setminus i_k) + v(i_k) \leq v(T)$
  - $\implies v(T) \leq v(S)$
- *Where do we need non-negativity in the proof?*
  - $v(\emptyset) > v(i)$ does not violate **weak superadditivity**
  - actually, $v(i) \geq 0$ for $i \in N \implies$ **monotonicity**

Classes of games VI.

Positive cooperative games are **monotone**.

- $v(i) = m^v(i) \geq 0$

Big picture of classes of games

Do we need games with negative values?

Strategic equivalence

Two game (N, v) and (N, w) are strategically equivalent if there is $\alpha > 0$ and $\beta \in \mathbb{R}^n$ such that

$$w(S) = \alpha v(S) + \beta(S).$$

- $\phi(w) = \alpha\phi(v) + \beta$
- $\eta(w) \neq \alpha\eta(v) + \beta$

Classes of games VII.

Weakly superadditive cooperative games are **essential**.

- $v(1) + v(2) \leq v(\{1,2\})$
- $v(\{1,2\}) + v(3) \leq v(\{1,2,3\})$
- $\vdots$
- $v(\{1,\ldots,n-1\}) + v(n) \leq v(N)$

Classes of games VIII.

There are **monotone** cooperative games which are not **essential**.

Example: $N = \{1, 2\}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$v(S)$	0	1	1	1

- $v(1) \leq v(\{1, 2\})$
- $v(2) \leq v(\{1, 2\})$
- $v(\emptyset) \leq v(1)$
- $v(\emptyset) \leq v(2)$
- $v(1) + v(2) = 2 \not\leq 1 = v(\{1, 2\})$

1. $\eta(v) \neq \emptyset \iff (N, v)$ is **essential**
   - *But what are **essential** games?*
   - E.g. **weakly superadditive** games
2. $\mathcal{C}(v) \neq \emptyset \iff (N, v)$ is **balanced**
   - *But what are **balanced** games?*
   - E.g. **convex** games
3. $\eta(v) \subseteq \mathcal{C}(v) \subseteq \mathcal{K}(v) \subseteq (v)$
   - *But when does $=$ or $\subsetneq$ hold?*
   - E.g. **convex** games

Convex games

A cooperative game (N, v) is **convex** if

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T).$$

Alternative characterisation

Bigger coalition have bigger marginal contribution

Characterisation of convex games

A cooperative game (N, v) is convex if and only if

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$$

for $S \subseteq T \subseteq N \setminus i$.

Proof: $\implies$

- $v(A) + v(B) \leq v(A \cap B) + v(A \cup B)$
  - $A = S \cup i$
  - $B = T$
- $v(S \cup i) + v(T) \leq v((S \cup i) \cap T) + v((S \cup i) \cup T)$
  - $(S \cup i) \cap T = S$
  - $(S \cup i) \cup T = T \cup i$
- $v(S \cup i) + v(T) \leq v(S) + v(T \cup i)$

Bigger coalition have bigger marginal contribution

Characterisation of convex games

A cooperative game (N, v) is convex if and only if
$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T).$$
for $S \subseteq T \subseteq N \setminus i$.

Proof: $\Longleftarrow$: $v(A) + v(B) \leq v(A \cap B) + v(A \cup B)$

$$v((A \cap B) \cup i_1) - v(A \cap B) \leq v(B \cup i_1) - v(B)$$
$$v((A \cap B) \cup i_1 \cup i_2) - v((A \cap B) \cup i_1) \leq v(B \cup i_1 \cup i_2) - v(B \cup i_1)$$
$$\vdots \leq \vdots$$
$$v(A) - v(A \setminus i_k) \leq v(B \cup A) - v((B \cup A) \setminus i_k)$$

- We add the following:
- $A \setminus B = \{i_1, \ldots, i_k\}$

Bigger coalition have bigger marginal contribution

Characterisation of convex games

A cooperative game (N, v) is convex if and only if

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$$

for $S \subseteq T \subseteq N \setminus i$.

Proof: $\Longleftarrow$

- Sečteme následujicí:
  - $A \setminus B = \{i_1, \ldots, i_k\}$
  - $v((A \cap B) \cup i_1) - v(A \cap B) \leq v(B \cup i_1) - v(B)$
  - $v((A \cap B) \cup i_1 \cup i_2) - v((A \cap B) \cup i_1) \leq v(B \cup i_1 \cup i_2) - v(B \cup i_1)$
  - $\vdots$
  - $v(A) - v(A \setminus i_k) \leq v(B \cup A) - v((B \cup A) \setminus i_k)$

- For (N, v), we have $\mathcal{C}(v) \subseteq \mathcal{W}(v)$

Core is the Weber set

For a **convex** cooperative game (N, v), it holds $\mathcal{C}(v) = \mathcal{W}(v)$.

Proof: We show only $m_v^{id} \in \mathcal{C}(v)$

1. $m_v^{id}(N) = \sum_{i \in N} v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\}) = v(N)$
2. $m_v^{id}(S) \geq v(S)$
   - $S = \{s_1, s_2, \ldots, s_s\}$
     - $s_1 < s_2 < \cdots < s_s$
   - $v(\{1, \ldots, s_i\}) - v(\{1, \ldots, s_{i-1}\}) \geq v(\{s_1, \ldots, s_i\}) - v(\{s_1, \ldots, s_{i-1}\})$
     - $\{s_1, s_2, \ldots, s_k\} \subseteq \{1, \ldots, s_k\}$ for $k \leq s$
   - $m_v^{id}(S) = \sum_{s_i \in S} v(\{1, \ldots, s_i\}) - v(\{1, \ldots, s_{i-1}\})$
   - $m_v^{id}(S) \geq \sum_{s_i \in S} v(\{s_1, \ldots, s_i\}) - v(\{s_1, \ldots, s_{i-1}\}) = v(S)$

Equality implies convexity

If it holds $\mathcal{C}(v) = \mathcal{W}(v)$ for a coopertaive game (N, v), it follows (N, v) is **convex**.

Proof:

- $N = \{\underbrace{i_1, \ldots, i_k}_{S \cap T}, \underbrace{i_{k+1}, \ldots, i_\ell}_{T \setminus S}, \underbrace{i_{\ell+1}, \ldots, i_o}_{S \setminus T}, \underbrace{i_{o+1}, \ldots, i_p}_{N \setminus (S \cup T)}\}$
- $\sigma(j) = i_j$
- $m_v^\sigma(S) \geq v(S)$
- $v(S) \leq \sum_{i \in S} (m_v^\sigma)_i = \sum_{j=1}^{k} (m_v^\sigma)_{i_j} + \sum_{j=\ell+1}^{o} (m_v^\sigma)_{i_j} =$
- $= v(i_1, \ldots, i_k) + v(i_1, \ldots, i_o) - v(i_1, \ldots, i_\ell) =$
- $= v(S \cap T) + v(S \cup T) - v(T)$

Shapley is the centre of gravity of the core

- $\phi(v) = \sum_{\sigma \in \Sigma_n} \frac{m_v^\sigma}{n!}$
- For **convex** game (N, v):
    1. $\phi(v) \in \mathcal{C}(v)$
    2. $\phi(v)$ is the centre of gravity of $\mathcal{C}(v)$

- in general, $\eta(v) \subseteq \mathcal{K}(v)$

Kernel of convex games

For a cooperative game (N, v), it holds

$$\mathcal{K}(v) = \eta(v).$$

Proof: Shapley (1972) - *too long*

Classes of games

Classes of **positive**, **convex**, **superadditive** and **weakly superadditive** form a hierarchy. When non-negative, they are all **monotone**. **Weakly superadditive** games have non-empty nucleolus $\eta(v)$. Further, **convex** cooperative games have a non-empty core, which is always equal to the Weber set. As a consequence, the Shapley value is always contained by the core and forms its centre of gravity. Last, it holds $\eta(v) = \mathcal{K}(v)$ for **convex** cooperative games.