Cooperative game theory

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Classes of games

Motivation

1. $\mathcal{C}(v) \neq \emptyset \iff (N, v)$ is **balanced**
   - *But what are **balanced** games?*
2. $\eta(v) \neq \emptyset \iff (N, v)$ is **essential**
   - *But what are **essential** games?*
3. $\eta(v) \subseteq \mathcal{C}(v) \subseteq \mathcal{K}(v) \subseteq (v)$
   - *But when does $=$ or $\subsetneq$ hold?*

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7. **balanced** if $\mathcal{C}(v) \neq \emptyset$.

What is the relation between the classes?

Relations between classes I.

Balanced cooperative games are **essential**.

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  - $\emptyset \neq \mathcal{C}(v) \subseteq \mathcal{I}(v)$

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- $0 \leq \sum_{X \subseteq S \cup T, X \not\subseteq S, X \not\subseteq T} m^v(X)$

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- $0 \leq \sum_{X \subseteq S \cup T, X \not\subseteq S, X \not\subseteq T} m^v(X)$
  - holds since $m^v(X) \geq 0$ for every $X \subseteq N$

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Superadditive cooperative games are **weakly superadditive**.

- $v(S) + v(T) \leq v(S \cup T)$, $S \cap T = \emptyset$
- $T = \{i\}$
  - $v(S) + v(i) \leq v(S \cup i)$

Monotonic games

Example: $N = \{1, 2\}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1, 2\}$
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1. $v(1) + v(2) \leq v(\{1,2\})$
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   - $\implies$ **weakly superadditive**
2. $v(\emptyset) \not\leq v(\{1\})$
   - $\implies$ **not monotone**

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- **Negative values** are the problem!

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  3. $\vdots$
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- $v(i) = m^v(i) \geq 0$

Big picture of classes of games

Strategic equivalence

Two game (N, v) and (N, w) are strategically equivalent if there is $\alpha > 0$ and $\beta \in \mathbb{R}^n$ such that

$$w(S) = \alpha v(S) + \beta(S).$$

- $\phi(w) = \alpha\phi(v) + \beta$
- $\eta(w) \neq \alpha\eta(v) + \beta$

Classes of games VII.

Weakly superadditive cooperative games are **essential**.

- $v(1) + v(2) \leq v(\{1, 2\})$
- $v(\{1, 2\}) + v(3) \leq v(\{1, 2, 3\})$
- $\vdots$
- $v(\{1, \ldots, n-1\}) + v(n) \leq v(N)$

Classes of games VIII.

There are **monotone** cooperative games which are not **essential**.

Example: $N = \{1, 2\}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$v(S)$	0	1	1	1

- $v(1) \leq v(\{1,2\})$
- $v(2) \leq v(\{1,2\})$
- $v(\emptyset) \leq v(1)$
- $v(\emptyset) \leq v(2)$
- $v(1) + v(2) = 2 \not\leq 1 = v(\{1,2\})$

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   - *But what are **essential** games?*
   - E.g. **weakly superadditive** games
2. $\mathcal{C}(v) \neq \emptyset \iff (N, v)$ is **balanced**
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Convex games

A cooperative game (N, v) is **convex** if

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T).$$

Bigger coalition have bigger marginal contribution

Characterisation of convex games

A cooperative game (N, v) is convex if and only if

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$$

for $S \subseteq T \subseteq N \setminus i$.

Proof: $\implies$

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- $v(A) + v(B) \leq v(A \cap B) + v(A \cup B)$
  - $A = S \cup i$
  - $B = T$

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- $v(S \cup i) + v(T) \leq v((S \cup i) \cap T) + v((S \cup i) \cup T)$
  - $(S \cup i) \cap T = S$
  - $(S \cup i) \cup T = T \cup i$

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- $v(S \cup i) + v(T) \leq v(S) + v(T \cup i)$

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Proof: $\Longleftarrow$: $v(A) + v(B) \leq v(A \cap B) + v(A \cup B)$

$$v((A \cap B) \cup i_1) - v(A \cap B) \leq v(B \cup i_1) - v(B)$$

$$v((A \cap B) \cup i_1 \cup i_2) - v((A \cap B) \cup i_1) \leq v(B \cup i_1 \cup i_2) - v(B \cup i_1)$$

$$\vdots \leq \vdots$$

$$v(A) - v(A \setminus i_k) \leq v(B \cup A) - v((B \cup A) \setminus i_k)$$

- $A \setminus B = \{i_1, \ldots, i_k\}$

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$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T).$$

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Proof: $\Longleftarrow$: $v(A) + v(B) \leq v(A \cap B) + v(A \cup B)$

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$$v((A \cap B) \cup i_1 \cup i_2) - v((A \cap B) \cup i_1) \leq v(B \cup i_1 \cup i_2) - v(B \cup i_1)$$

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$$v(A) - v(A \setminus i_k) \leq v(B \cup A) - v((B \cup A) \setminus i_k)$$

- We add the following:
- $A \setminus B = \{i_1, \ldots, i_k\}$

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Proof: $\Longleftarrow$

- Sečteme následující:
  - $A \setminus B = \{i_1, \ldots, i_k\}$

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for $S \subseteq T \subseteq N \setminus i$.

Proof: $\Longleftarrow$

- Sečteme následujicí:
  - $A \setminus B = \{i_1, \ldots, i_k\}$
  - $v((A \cap B) \cup i_1) - v(A \cap B) \leq v(B \cup i_1) - v(B)$
  - $v((A \cap B) \cup i_1 \cup i_2) - v((A \cap B) \cup i_1) \leq v(B \cup i_1 \cup i_2) - v(B \cup i_1)$
  - $\vdots$
  - $v(A) - v(A \setminus i_k) \leq v(B \cup A) - v((B \cup A) \setminus i_k)$

- For (N, v), we have $\mathcal{C}(v) \subseteq \mathcal{W}(v)$

Core is the Weber set

For a **convex** cooperative game (N, v), it holds $\mathcal{C}(v) = \mathcal{W}(v)$.

Proof: We show only $m_v^{id} \in \mathcal{C}(v)$

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  1. $\phi(v) \in \mathcal{C}(v)$
  2. $\phi(v)$ is the centre of gravity of $\mathcal{C}(v)$

- in general, $\eta(v) \subseteq \mathcal{K}(v)$

Kernel of convex games

For a cooperative game (N, v), it holds

$$\mathcal{K}(v) = \eta(v).$$

Proof: Shapley (1972) - *too long*

Classes of games

Classes of **positive**, **convex**, **superadditive** and **weakly superadditive** form a hierarchy. When non-negative, they are all **monotone**. **Weakly superadditive** games have non-empty nucleolus $\eta(v)$. Further, **convex** cooperative games have a non-empty core, which is always equal to the Weber set. As a consequence, the Shapley value is always contained by the core and forms its centre of gravity. Last, it holds $\eta(v) = \mathcal{K}(v)$ for **convex** cooperative games.