COOPERATIVE GAME THEORY

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THE BARGAINING SET AND THE KERNEL

MOTIVATION

- What if we desire the *stability* the core offers, but $\mathcal{C}(v) = \emptyset$?

1. we have the nucleolus $\eta(v)$
2. What if we want more options?
    - We need to *relax* the stability requirements

THE BARGAINING PROCESS

- An argument of player i against payoff x
  - *I get too little in the imputation x, and agent j gets too much! I can form a coalition that excludes j in which some members benefit and all members are at least as well-off as in x.*

Objection

An objection of i against j to x is a pair (S, y) where

- $S \subseteq N$, $i \in S$, $j \notin S$,
- $y \in \mathbb{R}^s$, $y(S) \leq v(S)$ (y is feasible for S)
- $\forall k \in S, y_k \geq x_k$ and $y_i > x_i$ (nobody is worse off and i gains)

- **Goal:** To obtain a side payment from j to i

THE BARGAINING PROCESS

- Answer to an argument
  - *I can form a coalition that excludes agent i in which all agents are at least as well off as in x, and as well off as in the payoff proposed by i for those who were offered to join i in the argument.*

Counter-objection

A counter-objection to (S, y) is a pair (T, z) where

- $T \subseteq N$, $j \in T$, $i \notin T$,
- $z \in \mathbb{R}^s$, $z(T) \leq v(T)$ (z is feasible for T)
- $\forall k \in T, z_k \geq x_k$ (nobody is worse off)
- $\forall k \in T \cap S, z_k \geq y_k$ ($k \in Q \cap P$ get at least as much as in (S, y))

- **Goal:** To show j can protect x_j from the objection of i

STABILITY OF THE PRE-BARGAINING SET

Stability

For a cooperative game (N, v), vector $x \in \mathbb{R}^n$ is stable if for each objection at x there is a counter-objection.

The bargaining set

For a cooperative game (N, v), the bargaining set $\mathcal{BS}(v)$ is defined as

$$\mathcal{BS}(v) := \{x \in \mathcal{I}(v) \mid x \text{ is stable}\}.$$

RELATION BETWEEN THE STABILITY AND THE CORE STABILITY

Core is a subset of the bargaining set

For a cooperative game (N, v), it holds

$$\mathcal{C}(v) \subseteq \mathcal{BS}(v).$$

Proof: *There are no objections for* $x \in \mathcal{C}(v)$*!*

- $x(S) \geq v(S)$
- objection (S, y) satisfies $y(S) > x(S)$ and $y(S) \leq v(S)$
- $v(S) \geq y(S) > x(S) \geq v(S)$

OBJECTIONS HIDDEN BEHIND COALITIONS

Idea: Players pretend to care about the welfare of coalitions.

- *S is a coalition that contains i, excludes j and which sacrifices too much (or gains too little).*

Objection

A coalition $S \subseteq N$ is an <u>objection of i against j to x</u> if $i \in S$, $j \notin S$ and $x_j > v(j)$.

- *Player i's demand is not justified: T is a coalition that contains j and excludes i and that sacrifices even more (or gains even less)*

Counter-Objection

A coalition $T \subseteq N$ is a <u>counter-objection to the objection P of i against j</u> if $j \in T$, $i \notin T$ and $e(T, x) \geq e(S, x)$.

THE KERNEL

The kernel

For a cooperative game (N, v), the kernel $\mathcal{K}(v)$ is defined as

$$\mathcal{K}(v) = \left\{ x \in \mathcal{I}(v) \,\middle|\, \begin{array}{l} \forall S \text{ objection of } i \text{ over } j \text{ to } x, \\ \exists T \text{ a counter-objection of } j \text{ to } S. \end{array} \right\}$$

ALTERNATIVE DEFINITION OF THE KERNEL

The kernel

For a cooperative game (N, v), the kernel $\mathcal{K}(v)$ is defined as

$$\mathcal{K}(v) = \left\{ x \in \mathcal{I}(v) \middle| \begin{array}{l} \forall S \text{ objection of } i \text{ over } j \text{ to } x, \\ \exists T \text{ a counter-objection of } j \text{ to } S \end{array} \right\}.$$

- denote $S_{ij} \subseteq N$ such that $i \in S_{ij}$ and $j \notin S_{ij}$

1. $x_j = v(j)$ ... i does not have an objection against j
2. $x_j > v(j)$ ... every $S_{ij} \subseteq N$ is an objection
    - every S_{ji} satisfying $e(S_{ji}, x) \geq e(S_{ij}, x)$ is a counter-objection

- player i is *safe* against j if
    1. $x_j = v(j)$, or
    2. $x_j > v(j)$ and $\max\limits_{S_{ji} \subseteq N} e(S_{ji}, x) \geq \max\limits_{S_{ij} \subseteq N} e(S, x)$
- $s_{ij}(x) := \max\limits_{S_{ij} \subseteq N} e(S_{ij}, x)$

RELATION BETWEEN THE KERNEL AND THE BARGAINING SET

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\mathcal{K}(v) \subseteq \mathcal{BS}(v)$.

Proof: $x \in \mathcal{K}(v) \implies x \in \mathcal{BS}(v)$

- (S_{ij}, y) ... objection of i against j to x
  - $y(S_{ij}) \leq v(S_{ij}), \forall k: \; y_k \geq x_k$ and $y_i > x_i \quad (y(S_{ij}) > x(S_{ij}))$
  - we choose $y(S_{ij}) = v(S_{ij})$
- we need (S_{ji}, z) s.t.
  - $z(S_{ji}) \leq v(S_{ji})$,
    - $\forall k \in Q: \; z_k \geq x_k$
    - $\forall k \in Q \cap P: \; z_k \geq y_k$

1. $x_j = v(j)$
   - choose $S_{ji} = \{j\}$ and $y_j = v(j)$

RELATION BETWEEN THE KERNEL AND THE BARGAINING SET

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Proof: $x \in \mathcal{K}(v) \implies x \in \mathcal{BS}(v)$

- (S_{ij}, y) ... objection of i against j to x
  - $y(S_{ij}) \leq v(S_{ij}), \forall k: \; y_k \geq x_k$ and $y_i > x_i \quad (y(S_{ij}) > x(S_{ij}))$
  - we choose $y(S_{ij}) = v(S_{ij})$
- we need (S_{ji}, z) s.t.
  - $z(S_{ji}) \leq v(S_{ji})$,
    - $\forall k \in Q: \; z_k \geq x_k$
    - $\forall k \in Q \cap P: \; z_k \geq y_k$

2. $x_j > v(j)$
   - choose $S_{ji}^* \subseteq N$ s.t.
   - $v(S_{ji}^*) - x(S_{ji}^*) = s_{ji}(x)$

RELATION BETWEEN THE KERNEL AND THE BARGAINING SET

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\mathcal{K}(v) \subseteq \mathcal{BS}(v)$.

Proof: $x \in \mathcal{K}(v) \implies x \in \mathcal{BS}(v)$

- (S_{ij}, y) ... objection of i against j to x
  - $y(S_{ij}) \leq v(S_{ij}), \forall k : y_k \geq x_k$ and $y_i > x_i \quad (y(S_{ij}) > x(S_{ij}))$
- we need (S_{ji}, z) s.t.
  - $z(S_{ji}) \leq v(S_{ji}),$
    - $\forall k \in Q : \; z_k \geq x_k, \forall k \in Q \cap P : \; z_k \geq y_k$

2. $x_j > v(j)$
   - $v(S^*_{ji}) - x(S^*_{ji}) \geq y(S_{ij}) - x(S_{ij})$
   - $v(S^*_{ji}) \geq y(S_{ij}) + x(S^*_{ji}) - x(S_{ij})$
   - $= y(S_{ij} \cap S^*_{ji}) + y(S_{ij} \setminus S^*_{ji}) + x(S^*_{ji} \setminus S_{ij}) - x(S_{ij} \setminus S^*_{ji})$
   - $> y(S_{ij} \cap S^*_{ji}) + x(S^*_{ji} \setminus S_{ij})$
     - $y(S_{ij} \setminus S^*_{ji}) > x(S_{ij} \setminus S^*_{ji})$ since $i \in S_{ij} \setminus S^*_{ji}$

RELATION BETWEEN THE KERNEL AND THE BARGAINING SET

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\mathcal{K}(v) \subseteq \mathcal{BS}(v)$.

Proof: $x \in \mathcal{K}(v) \implies x \in \mathcal{BS}(v)$

- (S_{ij}, y) ... objection of i against j to x
- we need (S_{ji}, z) s.t.
  - $z(S_{ji}) \leq v(S_{ji})$,
    - $\forall k \in Q : \; z_k \geq x_k$,
    - $\forall k \in Q \cap P : \; z_k \geq y_k$

2. $x_j > v(j)$
   - $v(S^*_{ji}) > y(S^*_{ji} \cap S_{ij}) + x(S^*_{ji} \setminus S_{ij})$
   - $z_k := \begin{cases} x_k & \text{if } k \in S^*_{ji} \setminus S_{ij}, \\ y_k & \text{if } k \in S^*_{ji} \cap S_{ij} \end{cases}$

RELATION BETWEEN $\mathcal{K}(v)$ AND $\eta(v)$

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\eta(v) \subseteq \mathcal{K}(v)$.

Proof:

- $\exists i, j \in N$: $s_{ji}(x) > s_{ij}(x)$ and $x_i > v(i)$
- $y_k = \begin{cases} x_k & k \neq i, k \neq j, \\ x_k - \varepsilon & k = i, \\ x_k + \varepsilon & k = j. \end{cases}$
    - y ... reflects transfer of $\varepsilon > 0$ from i to j
    - choose ε :
        1. $x_i - \varepsilon = y_i > v(i)$
        2. $s_{ji}(y) > s_{ij}(y)$
- Goal: Show $\theta(y) \prec_{lex} \theta(x)$
    - $\implies x \notin \eta(v)$

RELATION BETWEEN $\mathcal{K}(v)$ AND $\eta(v)$

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\eta(v) \subseteq \mathcal{K}(v)$.

Proof: $x \notin \mathcal{K}(v) \implies x \notin \eta(v)$

- $\exists i, j \in N$: $s_{ji}(x) > s_{ij}(x)$ and $x_i > v(i)$
- y ... reflects transfer of $\varepsilon > 0$ from i to j
- $\theta(x) = (\bullet, \circ, \bullet, \bullet, \circ | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$
    - $\bullet$ ... $e(S_{ij}, x)$
    - $\bullet$ ... $e(S_{ji}, x)$
    - $\circ$ ... $e(S, x)$, $i, j \in S$ or $i, j \notin S$
    - $|$ ... divides entries with different values
        - $\circ | \bullet$, $\bullet$ ... $e(S, x) > e(S_{ji}, x) = e(S_{ij}, x)$

RELATION BETWEEN $\mathcal{K}(v)$ AND $\eta(v)$

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\eta(v) \subseteq \mathcal{K}(v)$.

Proof: $x \notin \mathcal{K}(v) \implies x \notin \eta(v)$

- $\exists i, j \in N$: $s_{ji}(x) > s_{ij}(x)$ and $x_i > v(i)$
- y ... reflects transfer of $\varepsilon > 0$ from i to j
- $\theta(x) = (\bullet, \circ, \bullet, \bullet, \circ | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$
- $\theta(y) = ?$
  - $e(S_{ij}, y) = v(S) - x(S) + \varepsilon > e(S_{ij}, x)$
  - $e(S_{ji}, y) = v(S) - x(S) - \varepsilon < e(S_{ij}, x)$
- $\theta(x) = (\bullet, \bullet, \circ, \circ, \bullet | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$
  - rearranging inside $| \ldots |$ does not changes anything
- $\theta(x) = (\circ, \circ, \circ, \circ, \bullet | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$
  - $s_{ji}(x) > s_{ij}(x) \iff \max_{S_{ji} \subseteq N} e(S_{ji}, x) > \max_{S_{ij} \subseteq N} e(S_{ij}, x)$

RELATION BETWEEN $\mathcal{K}(v)$ AND $\eta(v)$

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\eta(v) \subseteq \mathcal{K}(v)$.

Proof: $x \notin \mathcal{K}(v) \implies x \notin \eta(v)$

- $\exists i, j \in N$: $s_{ji}(x) > s_{ij}(x)$ and $x_i > v(i)$
- y ... reflects transfer of $\varepsilon > 0$ from i to j
- $\theta(x) = (\circ, \circ, \circ, \circ, \bullet | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$
- $\theta(y) = (\circ, \circ, \circ, \circ, \underset{-\varepsilon}{\bullet} | \overset{+\varepsilon}{\bullet}, \underset{-\varepsilon}{\bullet}, \underset{-\varepsilon}{\bullet} | \ldots | \overset{+\varepsilon}{\bullet}, \circ, \underset{-\varepsilon}{\bullet})$
  - **key:** we set $\varepsilon : s_{ji}(y) > s_{ij}(y) \iff \max e(S_{ji}, y) > \max e(S_{ij}, y)$
  - $\implies$ the order of first $\bullet, \bullet$ does not change
- $\theta(x) = (\circ, \circ, \circ, \circ, \bullet | \bullet, \ldots)$
- $\theta(y) = (\circ, \circ, \circ, \circ, | \bullet | \bullet, \ldots)$
- $\theta(y) \prec_{lex} \theta(x) \implies x \notin \eta(v)$

NON-EMPTYNESS OF $\mathcal{K}(v)$ AND $\mathcal{BS}(v)$

Relation of solution concepts

For a cooperative game (N, v), it holds

$$\eta(v) \subseteq \mathcal{C}(v) \subseteq \mathcal{K}(v) \subseteq \mathcal{BS}(v).$$

Following theorem is immediate.

Non-emptyness of $\mathcal{K}(v)$ and $\mathcal{BS}(v)$

For a cooperative game (N, v), if it holds $\mathcal{I}(v) \neq \emptyset$, then we have $\mathcal{K}(v) \neq \emptyset$ and $\mathcal{BS}(v) \neq \emptyset$.

NUCLEOLUS IN TERMS OF (COUNTER-)OBJECTIONS

Idea: Objections made by coalitions instead of players.

- *Our excess for coalition P is too large at x, payoff y reduces it.*

Objection

A pair (P, y), in which $P \subseteq N$ and $y \in \mathcal{I}(v)$ is an objection to x if $e(P, x) > e(P, y)$.

- *Our excess under y is larger than it was under x for coalition Q! Furthermore, our excess at y is larger than what your excess was at x!*

Counter-objection

A coalition (Q, y) is a counter-objection to the objection (P, y) when $e(Q, y) > e(Q, x)$ and $e(Q, y) > e(P, x)$.

NUCLEOLUS IN TERMS OF (COUNTER-)OBJECTIONS

Objection

A pair (P, y), in which $P \subseteq N$ and $y \in \mathcal{I}(v)$ is <u>an objection to x</u> if $e(P, x) > e(P, y)$.

Counter-objection

A coalition (Q, y) is <u>a counter-objection to the objection (P, y)</u> when $e(Q, y) > e(Q, x)$ and $e(Q, y) > e(P, x)$.

- $x \in \eta(v)$ is not stable
  - there is y: $e(P, x) > e(P, y)$
  - and for every $S \subseteq N$: $e(Q, y) \leq e(Q, x)$ or $e(Q, y) \leq e(P, x)$
  - $\implies \theta(y) \preceq_{lex} \theta(x) \implies x \notin \eta(v)$

NUCLEOLUS IN TERMS OF (COUNTER-)OBJECTIONS

Objection

A pair (P, y), in which $P \subseteq N$ and $y \in \mathcal{I}(v)$ is an objection to x if $e(P, x) > e(P, y)$.

Counter-objection

A coalition (Q, y) is a counter-objection to the objection (P, y) when $e(Q, y) > e(Q, x)$ and $e(Q, y) > e(P, x)$.

Nucleolus

For a cooperative game (N, v), the nucleolus $\eta(v)$ is defined as

$$\eta(v) = \{x \in \mathcal{I}(v) \mid x \text{ is stable.}\}.$$

SUMMARY

The bargaining set and the kernel

The bargaining set and the kernel are solution concepts, which relax the notion of **core stability**. The stability for both of these sets is defined through a bargaining process consisting of objections and counter-objections to payoff vectors. If each objection is covered by a counter-objection, the payoff vector is proclaimed **stable**. Both sets are generalisations of the nucleolus and when the core is non-empty, all the mentioned solution concepts together with the core form a hierarchy of stable solutions.