COOPERATIVE GAME THEORY

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THE BARGAINING SET AND THE KERNEL

MOTIVATION

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2. What if we want more options?
    - We need to *relax* the stability requirements

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Objection

An objection of *i* against *j* to *x* is a pair (S, y) where

- $S \subseteq N$, $i \in S$, $j \notin S$,
- $y \in \mathbb{R}^s$, $y(S) \leq v(S)$ (y is feasible for S)
- $\forall k \in S, y_k \geq x_k$ and $y_i > x_i$ (nobody is worse off and i gains)

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- **Goal:** To obtain a side payment from j to i

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- **Goal:** To show j can protect x_j from the objection of i

STABILITY OF THE PRE-BARGAINING SET

Stability

For a cooperative game (N, v), vector $x \in \mathbb{R}^n$ is stable if for each objection at x there is a counter-objection.

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The bargaining set

For a cooperative game (N, v), the bargaining set $\mathcal{BS}(v)$ is defined as

$$\mathcal{BS}(v) := \{x \in \mathcal{I}(v) \mid x \text{ is stable}\}.$$

RELATION BETWEEN THE STABILITY AND THE CORE STABILITY

Core is a subset of the bargaining set

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- $x(S) \geq v(S)$

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- $v(S) \geq y(S) > x(S) \geq v(S)$

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Counter-Objection

A coalition $T \subseteq N$ is a <u>counter-objection to the objection P of i against j</u> if $j \in T$, $i \notin T$ and $e(T, x) \geq e(S, x)$.

THE KERNEL

The kernel

For a cooperative game (N, v), the kernel $\mathcal{K}(v)$ is defined as

$$\mathcal{K}(v) = \left\{ x \in \mathcal{I}(v) \middle| \begin{array}{l} \forall S \text{ objection of } i \text{ over } j \text{ to } x, \\ \exists T \text{ a counter-objection of } j \text{ to } S. \end{array} \right\}$$

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    - $v(S_{ji}^*) - x(S_{ji}^*) = s_{ji}(x) \geq s_{ij}(x) \geq v(S_{ij}) - x(S_{ij}) = y(S_{ij}) - x(S_{ij})$

RELATION BETWEEN THE KERNEL AND THE BARGAINING SET

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\mathcal{K}(v) \subseteq \mathcal{BS}(v)$.

Proof: $x \in \mathcal{K}(v) \implies x \in \mathcal{BS}(v)$

- (S_{ij}, y) ... objection of i against j to x
  - $y(S_{ij}) \leq v(S_{ij})$, $\forall k : y_k \geq x_k$ and $y_i > x_i \quad (y(S_{ij}) > x(S_{ij}))$
- we need (S_{ji}, z) s.t.
  - $z(S_{ji}) \leq v(S_{ji})$,
    - $\forall k \in Q : \ z_k \geq x_k$, $\forall k \in Q \cap P : \ z_k \geq y_k$

2. $x_j > v(j)$
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2. $x_j > v(j)$
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   - $v(S^*_{ji}) \geq y(S_{ij}) + x(S^*_{ji}) - x(S_{ij})$

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2. $x_j > v(j)$
    - $v(S^*_{ji}) - x(S^*_{ji}) \geq y(S_{ij}) - x(S_{ij})$
    - $v(S^*_{ji}) \geq y(S_{ij}) + x(S^*_{ji}) - x(S_{ij})$
    - $= y(S_{ij} \cap S^*_{ji}) + y(S_{ij} \setminus S^*_{ji}) + x(S^*_{ji} \setminus S_{ij}) - x(S_{ij} \setminus S^*_{ji})$

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  - $z(S_{ji}) \leq v(S_{ji}),$
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   - $v(S_{ji}^*) \geq y(S_{ij}) + x(S_{ji}^*) - x(S_{ij})$
   - $= y(S_{ij} \cap S_{ji}^*) + y(S_{ij} \setminus S_{ji}^*) + x(S_{ji}^* \setminus S_{ij}) - x(S_{ij} \setminus S_{ji}^*)$
   - $> y(S_{ij} \cap S_{ji}^*) + x(S_{ji}^* \setminus S_{ij})$
     - $y(S_{ij} \setminus S_{ji}^*) > x(S_{ij} \setminus S_{ji}^*)$ since $i \in S_{ij} \setminus S_{ji}^*$

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2. $x_j > v(j)$
   - $v(S^*_{ji}) > y(S^*_{ji} \cap S_{ij}) + x(S^*_{ji} \setminus S_{ij}) = z(S^*_{ji})$
   - $z_k := \begin{cases} x_k & \text{if } k \in S^*_{ji} \setminus S_{ij}, \\ y_k & \text{if } k \in S^*_{ji} \cap S_{ij} \end{cases}$

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Proof:

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Proof:

- $\exists i, j \in N$: $s_{ji}(x) > s_{ij}(x)$ and $x_i > v(i)$
- $y_k = \begin{cases} x_k & k \neq i, k \neq j, \\ x_k - \varepsilon & k = i, \\ x_k + \varepsilon & k = j. \end{cases}$
  - y ... reflects transfer of $\varepsilon > 0$ from i to j

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    - y ... reflects transfer of $\varepsilon > 0$ from i to j
    - choose ε :
        1. $x_i - \varepsilon = y_i > v(i)$
        2. $s_{ji}(y) > s_{ij}(y)$

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    - choose ε :
        1. $x_i - \varepsilon = y_i > v(i)$
        2. $s_{ji}(y) > s_{ij}(y)$
- Goal: Show $\theta(y) \prec_{lex} \theta(x)$
    - $\implies x \notin \eta(v)$

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- $\theta(x) = (\bullet, \circ, \bullet, \bullet, \circ | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$

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  - $\bullet$ ... $e(S_{ij}, x)$

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- $\theta(x) = ({\color{blue}\bullet}, \circ, {\color{green}\bullet}, {\color{green}\bullet}, \circ {\color{red}|} {\color{blue}\bullet}, {\color{green}\bullet}, {\color{blue}\bullet} {\color{red}|} \ldots {\color{red}|} {\color{green}\bullet}, \circ, {\color{blue}\bullet})$
  - ${\color{green}\bullet}$ ... ${\color{green}e(S_{ij}, x)}$
  - ${\color{blue}\bullet}$ ... ${\color{blue}e(S_{ji}, x)}$

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    - $\bullet$ ... $e(S_{ij}, x)$
    - $\bullet$ ... $e(S_{ji}, x)$
    - $\circ$ ... $e(S, x)$, $i, j \in S$ or $i, j \notin S$

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  - $|$ ... divides entries with different values

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    - $|$ ... divides entries with different values
        - $\circ | \bullet$, $\bullet$ ... $e(S, x) > e(S_{ji}, x) = e(S_{ij}, x)$

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- $\theta(y) = ?$

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- $\theta(x) = (\bullet, \circ, \bullet, \bullet, \circ | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$
- $\theta(y) = ?$
  - $e(S_{ij}, y) = v(S) - x(S) + \varepsilon > e(S_{ij}, x)$

Relation between $\mathcal{K}(v)$ and $\eta(v)$

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For a cooperative game (N, v), it holds $\eta(v) \subseteq \mathcal{K}(v)$.

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- $\exists i, j \in N$: $s_{ji}(x) > s_{ij}(x)$ and $x_i > v(i)$
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- $\theta(x) = (\bullet, \circ, \bullet, \bullet, \circ | \bullet, \bullet, \bullet | \ldots | \bullet, \circ, \bullet)$
- $\theta(y) = ?$
  - $e(S_{ij}, y) = v(S) - x(S) + \varepsilon > e(S_{ij}, x)$
  - $e(S_{ji}, y) = v(S) - x(S) - \varepsilon < e(S_{ij}, x)$

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- $\theta(y) = ?$
    - $e(S_{ij}, y) = v(S) - x(S) + \varepsilon > e(S_{ij}, x)$
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- $\theta(y) = ?$
  - $e(S_{ij}, y) = v(S) - x(S) + \varepsilon > \textcolor{green}{e(S_{ij}, x)}$
  - $e(S_{ji}, y) = v(S) - x(S) - \varepsilon < \textcolor{blue}{e(S_{ij}, x)}$
- $\theta(x) = (\textcolor{green}{\bullet}, \textcolor{green}{\bullet}, \circ, \circ, \textcolor{blue}{\bullet} \textcolor{red}{|} \textcolor{green}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{blue}{\bullet} \textcolor{red}{|} \ldots \textcolor{red}{|} \textcolor{green}{\bullet}, \circ, \textcolor{blue}{\bullet})$
  - rearranging inside $\textcolor{red}{|} \ldots \textcolor{red}{|}$ does not changes anything

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  - $e(S_{ji}, y) = v(S) - x(S) - \varepsilon < e(S_{ij}, x)$
- $\theta(x) = (\bullet, \bullet, \circ, \circ, \bullet | \bullet, \bullet, \bullet | \dots | \bullet, \circ, \bullet)$
  - rearranging inside $| \dots |$ does not changes anything
- $\theta(x) = (\circ, \circ, \circ, \circ, \bullet | \bullet, \bullet, \bullet | \dots | \bullet, \circ, \bullet)$
  - $s_{ji}(x) > s_{ij}(x) \iff \max\limits_{S_{ji} \subseteq N} e(S_{ji}, x) > \max\limits_{S_{ij} \subseteq N} e(S_{ij}, x)$

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- $\theta(x) = (\circ, \circ, \circ, \circ, \bullet | \bullet, \bullet, \bullet | \dots | \bullet, \circ, \bullet)$
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  - **key:** we set $\varepsilon : s_{ji}(y) > s_{ij}(y) \iff \max e(S_{ji}, y) > \max e(S_{ij}, y)$

RELATION BETWEEN $\mathcal{K}(v)$ AND $\eta(v)$

The kernel is a subset of the bargaining set

For a cooperative game (N, v), it holds $\eta(v) \subseteq \mathcal{K}(v)$.

Proof: $x \notin \mathcal{K}(v) \implies x \notin \eta(v)$

- $\exists i, j \in N$: $s_{ji}(x) > s_{ij}(x)$ and $x_i > v(i)$
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- $\theta(y) = (\circ, \circ, \circ, \circ, {\color{red}|}{\color{blue}\bullet}{\color{red}|}{\color{green}\bullet}, \ldots)$
- $\theta(y) \prec_{lex} \theta(x) \implies x \notin \eta(v)$

NON-EMPTYNESS OF $\mathcal{K}(v)$ AND $\mathcal{BS}(v)$

Relation of solution concepts

For a cooperative game (N, v), it holds

$$\eta(v) \subseteq \mathcal{C}(v) \subseteq \mathcal{K}(v) \subseteq \mathcal{BS}(v).$$

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Non-emptyness of $\mathcal{K}(v)$ and $\mathcal{BS}(v)$

For a cooperative game (N, v), if it holds $\mathcal{I}(v) \neq \emptyset$, then we have $\mathcal{K}(v) \neq \emptyset$ and $\mathcal{BS}(v) \neq \emptyset$.

NUCLEOLUS IN TERMS OF (COUNTER-)OBJECTIONS

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A pair (P, y), in which $P \subseteq N$ and $y \in \mathcal{I}(v)$ is <u>an objection to x</u> if $e(P, x) > e(P, y)$.

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A coalition (Q, y) is a counter-objection to the objection (P, y) when $e(Q, y) > e(Q, x)$ and $e(Q, y) > e(P, x)$.

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Nucleolus in terms of (counter-)objections

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    - $\implies \theta(y) \preceq_{lex} \theta(x) \implies x \notin \eta(v)$

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Nucleolus

For a cooperative game (N, v), the nucleolus $\eta(v)$ is defined as

$$\eta(v) = \{x \in \mathcal{I}(v) \mid x \text{ is stable.}\}.$$

SUMMARY

The bargaining set and the kernel

The bargaining set and the kernel are solution concepts, which relax the notion of **core stability**. The stability for both of these sets is defined through a bargaining process consisting of objections and counter-objections to payoff vectors. If each objection is covered by a counter-objection, the payoff vector is proclaimed **stable**. Both sets are generalisations of the nucleolus and when the core is non-empty, all the mentioned solution concepts together with the core form a hierarchy of stable solutions.