Cooperative game theory

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March 3, 2023

The Shapley value and the Weber set

$x \in \mathcal{C}(v)$ may be *unfair*

- N ... $2k+1$ players
  - k have left shoe
  - $k+1$ have right shoe
- $v(S) = \min\{S_\ell, S_p\} \cdot 1000$
  - S_ℓ ... number of players in S with left shoe
  - S_p ... number of players in S with right shoe
  - $v(N) = k \cdot 1000$
  - $\mathcal{C}(v) = \{x\}$:
  - $x_i = \begin{cases} 1000 & i \in N_\ell \\ 0 & i \in N_p \end{cases}$

We want *fair* solution $\implies$ the Shapley value

- If $f(v): \Gamma^n \to \mathbb{R}^n$ is a single-point solution concepts satisfying

1. (AXIOM OF EFFICIENCE)
   - $\sum_{i \in N} f_i(v) = v(N)$
   - *All profit is shared among the players.*
2. (AXIOM OF SYMMETRY)
   - $\forall i, j \in N, (\forall S \subseteq N \setminus \{i, j\} : v(S \cup i) = v(S \cup j)) \implies f_i(v) = f_j(v)$
   - *Same value, same payment.*
3. (AXIOM OF NULL PLAYER)
   - $\forall i \in N \; (\forall S \subseteq N : v(S) = v(S \cup i)) \implies f_i(v) = 0$
   - *No pain no gain.*
4. (AXIOM OF ADDITIVITY)
   - $\forall v, w \in \Gamma^n : f(v + w) = f(v) + f(w)$
   - *Division to simple games lead to same payments.*

- then it is **uniquely determined!**

The Shapley value

The Shapley value $\phi\colon \Gamma^n \to \mathbb{R}^n$ is a single-point solution concept satisfying the following axioms:
EFFICIENCE, SYMMETRIE, NULL PLAYER ad ADDITIVITY.

- How to determine $\phi(v)$?
- **important:** we use additivity and the properties of Γ^n
  - Γ^n forms a vector space

Vector space Γ^n

Two faces of game (N, v):

1. $v : 2^N \to \mathbb{R}$
   - $v(\emptyset) = 0$
   - Γ^n ... set of cooperative games
2. $v \in \mathbb{R}^{2^n}$
   - $v \in \mathbb{R}^{2^n - 1}$
   - $(\mathbb{R}^{2^n - 1}, +, \cdot)$ ... *vector space of cooperative games*

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Vector space Γ^n

The set of cooperative games Γ^n forms a vector space, which is isomorphic to standard vector space of dimension $2^n - 1$.

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- How to determine $\phi(v)$?
- **important:** we use additivity and the properties of Γ^n
  - Γ^n forms a vector space
- $v = \sum_{S\subseteq N} \alpha_S b_S$
  - (N, b_S) ... basis game
- $\phi(v) = \phi(\sum_{S\subseteq N} \alpha_S b_S) = \sum_{S\subseteq N} \phi(\alpha_S b_S)$
  - follows from additivity

1. We determine $\phi(\alpha_S b_S)$
2. we sum over $\sum_{S\subseteq N}$

- **Key question:** What basis of Γ^n to choose?

Canonical basis (N, e_S) for $S \subseteq N$:

- $e_S(T) = \begin{cases} 1 & T = S, \\ 0 & T \neq S. \end{cases}$
- $v = \sum_{S \subseteq N} v(S) e_S$

Canonical basis (N, e_S) for $S \subseteq N$:

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Why is the canonical basis not suitable?

- $\phi(v) = \sum_{S \subseteq N} \phi(v(S) e_S)$

1. $e_S(N) = 0$ if $S \neq N$
2. $\phi_i(e_S) = \begin{cases} \frac{(s-1)!(n-s-2)!}{n!} & i \in S, \\ -\frac{s!(n-s-1)!}{n!} & i \notin S. \end{cases}$
   - difficult to compute

- The Shapley value ϕ satisfies the *CARRIER AXIOM*
  - S is the **carrier** of (N, v)
    - $\forall T \subseteq N : v(T) = v(S \cap T)$

The Shapley value satisfies CARRIER AXIOM

If $S \subseteq N$ is the carrier of cooperative game (N, v), then it holds $\sum_{i \in S} \phi_i(v) = v(N)$.

Proof:

- $T = T_S \cup T_{\overline{S}}$
  - $T_S = T \cap S$, $T_{\overline{S}} = T \setminus T_S$
- $v(T) = v(T_S)$
  - $v(T) = v(T_S \cup T_{\overline{S}}) = v(T_S)$
- $i \notin S$: $v(T \cup i) = v((T_S \cup T_{\overline{S}}) \cup i) = v(T_S) = v(T_S \cup T_{\overline{S}}) = v(T)$
  - $\implies$ i is null player $\implies \phi_i(v) = 0$
- $v(N) = \sum_{j \in N} \phi_j(v) = \sum_{j \in S} \phi_j(v) + \sum_{j \notin S} \phi_i(v) = \sum_{j \in S} \phi_j(v)$

Unanimity games

Carrier is the winning coalitions...

Basis from **unanimity games** (N, u_S) for $S \subseteq N$:

- $u_S(T) = \begin{cases} 1 & S \subseteq T, \\ 0 & S \not\subseteq T. \end{cases}$
- $v = \sum_{\emptyset \neq S \subseteq N} d_v(S) u_S$
  - $d_v(S)$ ... Harsanyi dividends
  - $v(S) = \sum_{T \subseteq S} d_v(T)$
- $d_v(i) = v(i)$
  - $v(i) = d_v(i)$
- $d_v(\{i,j\}) = v(\{i,j\}) - v(i) - v(j)$
  - $v(\{i,j\}) = d_v(i) + d_v(j) + d_v(\{i,j\}) = v(i) + v(j) + d_v(\{i,j\})$
- $d_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$
  - Iteratively + principle of inclusion and exclusion

The Shapley values of unanimity games

For $(N, \alpha u_S)$, where $S \subseteq N$, it holds

$$\phi_i(\alpha u_S) = \begin{cases} \frac{\alpha}{|S|} & i \in S, \\ 0 & i \notin S. \end{cases}$$

Proof:

- $i \notin S$, $T \subseteq N \setminus \{i\}$: $\alpha u_S(T \cup i) = \alpha u_S(T)$
  - $\alpha u_S(T) = \begin{cases} \alpha & \text{if } S \subseteq T \\ 0 & S \nsubseteq T \end{cases}$
  - $\alpha u_S(T \cup i) = \begin{cases} \alpha & \text{if } S \subseteq T \\ 0 & S \nsubseteq T \end{cases}$
- $\implies \phi_i(\alpha u_S) = 0$
- $\implies \sum_{k \in N} \phi_k(\alpha u_S) = \sum_{k \in S} \phi_k(\alpha u_S)$

The Shapley values of unanimity games

For $(N, \alpha u_S)$, where $S \subseteq N$, it holds

$$\phi_i(\alpha u_S) = \begin{cases} \frac{\alpha}{|S|} & i \in S, \\ 0 & i \notin S. \end{cases}$$

Proof:

- $i, j \in S$, $T \subseteq N \setminus \{i, j\}$
- $\alpha u_S(T \cup i) = \alpha u_S(T \cup j)$
  - $S \not\subseteq T \cup i$, $S \not\subseteq T \cup j$
  - $\alpha u_S(T \cup i) = 0 = \alpha u_S(T \cup j)$
- $\implies \phi_i(\alpha u_S) = \phi_j(\alpha u_S)$
- $\implies \sum_{k \in S} \phi_k(\alpha u_S) = |S| \cdot \phi_i(\alpha u_S)$
- $\alpha = \sum_{k \in N} \phi_k(\alpha u_S) = \sum_{k \in S} \phi_k(\alpha u_S) = |S| \cdot \phi_i(\alpha u_S)$

Deriving the formula for the Shapley value

- $\phi_i(v) = \phi_i(\sum_{S \subseteq N, i \in S} d_v(S) u_S) = \sum_{S \subseteq N, i \in S} \phi_i(d_v(S) u_S) =$
  - from additivity
- $= \sum_{S \subseteq N, i \in S} \frac{d_v(S)}{|S|} =$
  - already proved
- $= \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$
  - from the definition of $d_v(S)$
- It can be derived: $\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} (v(S \cup i) - v(S))$
  - find elegant proof and you pass th exam
    - Lloyd Shapley: *Easy to derive!*

The Shapley value

For a cooperative game (N, v), the Shapley value $\phi(v)$ is

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S \cup i) - v(S)\right)$$

- The players agree on the following procedure:
  1. *We form the grandcoalition N.*
  2. *We enter the coalition individually and randomly.*
  3. *When player i enters coalition S, he receives* $v(S \cup i) - v(S)$*.*
  - *What is player i's average profit?*
  - The Shapley value is the average over player *i*'s payments under this rule

Interpretation of the Shapley value

The Shapley value

For a cooperative game (N, v), the Shapley value $\phi(v)$ is

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S \cup i) - v(S) \right)$$

- The players agree on the following procedure:
  1. *We form the grandcoalition N.*
  2. *We enter the coalition individually and randomly.*
  3. *When player i enters coalition S, he receives* $v(S \cup i) - v(S)$.
- $s!(n-s-1)!$ ... number of situations, in which i enters S
- $n!$ ... number of all possible ways to construct N
- $\phi_i(v)$ **... the average value of player *i*'s payment**

Weber set

1. We fix one construction of N
   - $\sigma \in \Sigma_n$ ... represent it by permutation
   - $\sigma(i)$ ... the order, in which player i enters the coalition
2. we compute players' payments
   - m_v^σ ... **marginal vector**
   - $(m_v^\sigma)_i := v(S_{\sigma(i)} \cup i) - v(S_{\sigma(i)})$
     - $S_{\sigma(i)} := \{j \in N \mid \sigma(j) < \sigma(i)\}$ ... predecessors of i under σ
3. We consider all combinations
   - $\mathcal{W}(v) := conv\{m_v^\sigma \mid \sigma \in \Sigma_n\}$ ... **Weber set**

- It holds: $\phi(v) = \sum_{\sigma \in \Sigma_n} \frac{m_v^\sigma}{n!}$

Relation between the Shapley value and the Weber set

For a cooperative game (N, v), it holds

$$\phi(v) \in \mathcal{W}(v).$$

Moreover, $\phi(v)$ is the center of gravity of $\mathcal{W}(v)$.

The Weber set contains the core

For every cooperative game (N, v), it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.

Proof: By induction on $|N|$

- $|N| = 1$
  - $|\Sigma_1| = 1$
  - $\mathcal{W}(v) = \{m_v^{id}\}$
    - $m_v^{id} = v(1)$
  - $\mathcal{C}(v) = \{x\}$
    - $x(1) = v(1)$
    - $\mathcal{C}(v) = \{v(1)\}$
    - $\mathcal{C}(v) \subseteq \mathcal{W}(v)$

The Weber set contains the core

For every cooperative game (N, v), it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.

Proof: By induction on $|N|$

- We show for $x \in \mathcal{C}(v)$ on the *border* of $\mathcal{C}(v)$
  - $\mathcal{C}(v)$ is convex $\implies$ holds for all $x \in \mathcal{C}(v)$
- $\exists S \subseteq N : x(S) = v(S)$
  1. (S, v_S)
     - $v_S(T) = v(T)$ for $T \subseteq S$
  2. $(N \setminus S, w_S)$
     - $w_S(T) := v(S \cup T) - v(S)$ for $T \subseteq N \setminus S$
  1. $x^S \in \mathcal{C}(v_S)$
     - $x(T) \geq v(T) = v_S(T)$ for $T \subseteq S$
  2. $x^{N \setminus S} \in \mathcal{C}(w_S)$
     - $x(T) = x(T \cup S) - x(S) \geq v(T \cup S) - v(S) = w_S(T)$

The Weber set contains the core

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  - $\mathcal{C}(v)$ is convex $\implies$ holds for all $x \in \mathcal{C}(v)$
- $\exists S \subseteq N : x(S) = v(S)$
  1. $x^S \in \mathcal{C}(v_S) \subseteq \mathcal{W}(v_S)$
     - $x^S = \sum_{\sigma \in \Sigma_s} \alpha_\sigma m^\sigma_{v_S}$ ... convex combination of vertices of $\mathcal{W}(v_S)$
  2. $x^{N \setminus S} \in \mathcal{C}(w_S) \subseteq \mathcal{W}(w_S)$
     - $x^{N \setminus S} = \sum_{\tau \in \Sigma_{n-s}} \beta_\tau m^\tau_{w_S}$ ... convex combination of vertices of $\mathcal{W}(w_S)$
- $(\sigma, \tau) \in \Sigma_n$
  - $(\sigma, \tau)(i) = \begin{cases} \sigma(i) & i \in S \\ |S| + \tau(i) & i \in N \setminus S \end{cases}$
- $x = \sum_{(\sigma,\tau) \in \Sigma_n} \alpha_\sigma \beta_\tau m_v^{(\sigma,\tau)} \in \mathcal{W}(v)$

The Shapley value

The Shapley value is a single-point solution concept satisfying properties, which make it a fair solution. We can view it axiomatically, explicitly by different equivalent formulas and as an average payment of players under an agreed procedure. One of its multi-point generalisation is the Weber set; the Shapley value is its center of gravity.