Cooperative game theory

Martin Černý

kam.mff.cuni.cz/∼cerny
cerny@kam.mff.cuni.cz

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The Shapley value and the Weber set

$x \in \mathcal{C}(v)$ may be *unfair*

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- N ... $2k+1$ players
  - k have left shoe
  - $k+1$ have right shoe
- $v(S) = \min\{S_\ell, S_p\} \cdot 1000$
  - S_ℓ ... number of players in S with left shoe
  - S_p ... number of players in S with right shoe
  - $v(N) = k \cdot 1000$
  - $\mathcal{C}(v) = \{x\}$:
  - $x_i = \begin{cases} 1000 & i \in N_\ell \\ 0 & i \in N_p \end{cases}$

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We want *fair* solution $\implies$ the Shapley value

Fair solution

- If $f(v)\colon \Gamma^n \to \mathbb{R}^n$ is a single-point solution concepts satisfying

1. (AXIOM OF EFFICIENCE)
2. (AXIOM OF SYMMETRY)
3. (AXIOM OF NULL PLAYER)
4. (AXIOM OF ADDITIVITY)

- then it is **uniquely determined!**

Fair solution

- If $f(v)\colon \Gamma^n \to \mathbb{R}^n$ is a single-point solution concepts satisfying

1. (AXIOM OF EFFICIENCE)
   - *All profit is shared among the players.*
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- If $f(v)\colon \Gamma^n \to \mathbb{R}^n$ is a single-point solution concepts satisfying

1. (AXIOM OF EFFICIENCE)
   - $\sum_{i \in N} f_i(v) = v(N)$
   - *All profit is shared among the players.*
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   - *Same value, same payment.*
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2. (AXIOM OF SYMMETRY)
   - $\forall i, j \in N, (\forall S \subseteq N \setminus \{i, j\} : v(S \cup i) = v(S \cup j)) \implies f_i(v) = f_j(v)$
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1. (Axiom of efficience)
   - $\sum_{i \in N} f_i(v) = v(N)$
   - *All profit is shared among the players.*
2. (Axiom of symmetry)
   - $\forall i, j \in N, (\forall S \subseteq N \setminus \{i, j\} : v(S \cup i) = v(S \cup j)) \implies f_i(v) = f_j(v)$
   - *Same value, same payment.*
3. (Axiom of null player)
   - $\forall i \in N \ (\forall S \subseteq N : v(S) = v(S \cup i)) \implies f_i(v) = 0$
   - *No pain no gain.*
4. (Axiom of additivity)
   - *Division to simple games lead to same payments.*

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4. (AXIOM OF ADDITIVITY)
   - $\forall v, w \in \Gamma^n : f(v + w) = f(v) + f(w)$
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The Shapley value

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- **important:** we use additivity and the properties of Γ^n
  - Γ^n forms a vector space

Vector space Γ^n

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The set of cooperative games Γ^n forms a vector space, which is isomorphic to standard vector space of dimension $2^n - 1$.

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1. We determine $\phi(\alpha_S b_S)$
2. we sum over $\sum_{S \subseteq N}$

- **Key question:** What basis of Γ^n to choose?

Canonical basis (N, e_S) for $S \subseteq N$:

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Choice of basis of Γ^n

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1. $e_S(N) = 0$ if $S \neq N$
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   - difficult to compute

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  - S is the **carrier** of (N, v)
    - $\forall T \subseteq N : v(T) = v(S \cap T)$

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If $S \subseteq N$ is the carrier of cooperative game (N, v), then it holds $\sum_{i \in S} \phi_i(v) = v(N)$.

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Looking for a better basis

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  - $\implies$ i is null player $\implies \phi_i(v) = 0$
- $v(N) = \sum_{j \in N} \phi_j(v) = \sum_{j \in S} \phi_j(v) + \sum_{j \notin S} \phi_i(v) = \sum_{j \in S} \phi_j(v)$

Unanimity games

Carrier is the winning coalitions...

Basis from **unanimity games** (N, u_S) for $S \subseteq N$:

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- $d_v(\{i,j\}) = v(\{i,j\}) - v(i) - v(j)$
  - $v(\{i,j\}) = d_v(i) + d_v(j) + d_v(\{i,j\}) = v(i) + v(j) + d_v(\{i,j\})$

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  - $v(\{i,j\}) = d_v(i) + d_v(j) + d_v(\{i,j\}) = v(i) + v(j) + d_v(\{i,j\})$
- $d_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$
  - Iteratively + principle of inclusion and exclusion

The Shapley values of unanimity games

For $(N, \alpha u_S)$, where $S \subseteq N$, it holds

$$\phi_i(\alpha u_S) = \begin{cases} \frac{\alpha}{|S|} & i \in S, \\ 0 & i \notin S. \end{cases}$$

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- $\implies \phi_i(\alpha u_S) = 0$
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Deriving the formula for the Shapley value

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    - Lloyd Shapley: *Easy to derive!*

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  - *What is player i's average profit?*

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  - The Shapley value is the average over player *i*'s payments under this rule

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- $\phi_i(v)$ **... the average value of player** ***i*****'s payment**

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Relation between the Shapley value and the Weber set

For a cooperative game (N, v), it holds

$$\phi(v) \in \mathcal{W}(v).$$

Moreover, $\phi(v)$ is the center of gravity of $\mathcal{W}(v)$.

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For every cooperative game (N, v), it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.

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  1. $x^S \in \mathcal{C}(v_S)$
     - $x(T) \geq v(T) = v_S(T)$ for $T \subseteq S$

The Weber set contains the core

For every cooperative game (N, v), it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.

Proof: By induction on $|N|$

- We show for $x \in \mathcal{C}(v)$ on the *border* of $\mathcal{C}(v)$
  - $\mathcal{C}(v)$ is convex $\implies$ holds for all $x \in \mathcal{C}(v)$
- $\exists S \subseteq N : x(S) = v(S)$
  1. (S, v_S)
     - $v_S(T) = v(T)$ for $T \subseteq S$
  2. $(N \setminus S, w_S)$
     - $w_S(T) := v(S \cup T) - v(S)$ for $T \subseteq N \setminus S$
  1. $x^S \in \mathcal{C}(v_S)$
     - $x(T) \geq v(T) = v_S(T)$ for $T \subseteq S$
  2. $x^{N \setminus S} \in \mathcal{C}(w_S)$
     - $x(T) = x(T \cup S) - x(S) \geq v(T \cup S) - v(S) = w_S(T)$

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- $x = \sum_{(\sigma,\tau) \in \Sigma_n} \alpha_\sigma \beta_\tau m_v^{(\sigma,\tau)} \in \mathcal{W}(v)$

The Shapley value

The Shapley value is a single-point solution concept satisfying properties, which make it a fair solution. We can view it axiomatically, explicitly by different equivalent formulas and as an average payment of players under an agreed procedure. One of its multi-point generalisation is the Weber set; the Shapley value is its center of gravity.