Cooperative game theory

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Nucleolus

Motivation

- $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid x(S) \geq v(S) \text{ pro } S \subseteq N\}$ ... core of a game
- How to choose $x \in \mathcal{C}(v)$?

- $x(S) \geq v(S)$
- $x(S) + e = v(S)$
  - $e \leq 0$
  - size of e represents *dissatisfaction* of players over x
- $e(S, x) := v(S) - x(S)$ ... **excess** of coalition S over x
  - $\min e(S, x) \implies \max x(S)$
- $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid e(S, x) \leq 0 \text{ pro } S \subseteq N\}$

The excess

- $e(S,x) := v(S) - x(S)$ ... **excess** of coalition S over x
- $\theta(x) = (e(S_1,x), e(S_2,x), \ldots, e(S_{2^n-1},x))$ ... vector of excesses
  - $e(S_i,x) \geq e(S_j,x)$ pro $i \leq j$
- $\theta(x) \preceq_{lex} \theta(y)$ ... lexicographic ordering
  1. exists k: $\theta_k(x) < \theta_k(y)$
  2. for $\ell < k$: $\theta_\ell(x) = \theta_\ell(y)$

- $e(S,x) \coloneqq v(S) - x(S)$ ... **excess** of coalition S over x
- $\theta(x) = (e(S_1,x), e(S_2,x), \ldots, e(S_{2^n-1},x))$ ... vector of excesses
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- $\theta(x) \preceq_{lex} \theta(y)$ ... lexicographic ordering

Nucleolus

For a cooperative game (N,v), the **nucleolus** $\eta(v)$

$$\eta(v) \coloneqq \{x \in \mathcal{I}(v) \mid \theta(x) \preceq_{lex} \theta(y) \text{ for } y \in \mathcal{I}(v)\}.$$

Non-emptyness of the nucleolus

For a cooperative game (N, v), it holds
$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

Proof: Iterative usage of the Extreme value theorem

Extreme value theorem

A continuous function over compact set $\mathcal{M}$ attains its minimum. Moreover, the set of all minima forms a compact set $\mathcal{M}$.

- $e(S, \bullet)$ is continuous for $S \subseteq N \implies \theta_t(\bullet)$ is continuous
- $X_0 := \mathcal{I}(v)$
- $X_t := \{x \in X_{t-1} \mid \theta_t(x) \leq \theta_t(y) \text{ for } y \in X_{t-1}\}$
  - $t = 1, \ldots 2^n - 1$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: "$\theta(x) = \theta(y) \implies x = y$"

- $0 \leq \alpha \leq 1$
- $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1-\alpha)y) \succeq_{lex} \theta(x) = \theta(y)$
2. $\theta(\alpha x + (1-\alpha)y) \preceq_{lex} \theta(x) = \theta(y)$
3. $\theta(\alpha x + (1-\alpha)y) = \theta(x) = \theta(y)$
4. $x = y$

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1. $\theta(\alpha x + (1 - \alpha)y) \succeq_{lex} \theta(x) = \theta(y)$
   - from definition of $\eta(v)$ and $x, y \in \eta(v)$
2. $\theta(\alpha x + (1 - \alpha)y) \preceq_{lex} \theta(x) = \theta(y)$
3. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$
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2. $\theta(\alpha x + (1-\alpha)y) \preceq_{lex} \theta(x) = \theta(y)$
   - $\theta(\alpha x + (1-\alpha)y) \preceq_{lex} \alpha\theta(x) + (1-\alpha)\theta(y)$
   - home exercise
3. $\theta(\alpha x + (1-\alpha)y) = \theta(x) = \theta(y)$
4. $x = y$

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3. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$
   - $\succeq_{lex}$ is antisymmetric ordering
4. $x = y$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: "$\theta(x) = \theta(y) \implies x = y$"

- $0 < \alpha < 1$
- $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1-\alpha)y) = \theta(x) = \theta(y)$
2. $x = y$
   - $\theta(\alpha x + (1-\alpha)y) = \alpha a + (1-\alpha)b$
     - $S_1, S_2, \ldots, S_{2^n-1}$ ... pořadí koalic v $\theta(\alpha x + (1-\alpha)y)$
     - $a = (e(S_1, x), e(S_2, x), \ldots, e(S_{2^n-1}, x))$
     - $b = (e(S_1, y), e(S_2, y), \ldots, e(S_{2^n-1}, y))$
   - $\theta(x) = a$, $\theta(y) = b$
     - $\alpha = 1$, $\alpha = 0$
   - $e(S_i, x) = e(S_i, y)$ pro $i = 1, \ldots, 2^n - 1$
     - $e(\{i\}, x) = e(\{i\}, y)$
     - $v(i) - x_i = v(i) - y_i$
     - $x_i = y_i$

Nucleolus is a core selector

For a cooperative game (N, v), if it holds $\mathcal{C}(v) \neq \emptyset$ then $\eta(v) \subseteq \mathcal{C}(v)$.

Proof: $x \notin \mathcal{C}(v)$ *in contradiction with lexicographical minimality*

- $x \in \eta(v) \setminus \mathcal{C}(v) \implies \exists S \subseteq N : e(S, x) > 0$
- pro $y \in \mathcal{C}(v) : e(S, y) \leq 0$
- $\theta(x) \succ_{lex} \theta(y)$
  - $\theta_1(x) > \theta_1(y)$

- We can use the proof of non-emptyness
  - $X_0 := \mathcal{I}(v)$
  - $X_t := \{x \in X_{t-1} \mid \theta_t(y) \geq \theta_t(x) \text{ pro } y \in X_{t-1}\}$
    - $t = 1, \ldots 2^n - 1$
- $$LP(0) = \begin{cases} \min\limits_{x \in X_0, \alpha_0 \in \mathbb{R}} & \alpha_0 \\ \text{subject to} & \alpha_0 \geq x(S) - v(S) \\ & \forall S \in 2^N \setminus \{\emptyset\} \end{cases}$$
  - X_1 ... set of optimal solutions $LP(0)$
  - $\mathcal{B}_1 = \{S \subseteq 2^N \setminus \{\emptyset\} \mid e(S, x) = \alpha_0 \text{ pro } \forall x \in X_1\}$
- $$LP(t) = \begin{cases} \min\limits_{x \in X_t, \alpha_t \in \mathbb{R}} & \alpha_t \\ \text{subject to} & \alpha_t \geq x(S) - v(S) \\ & \forall \emptyset \neq S \in 2^N \setminus \mathcal{B}_t \end{cases}$$
  - X_{t+1} ... set of optimal solutions $LP(t)$
  - $\mathcal{B}_{t+1} = \{S \in 2^n \setminus (\mathcal{B}_t \cup \{\emptyset\}) \mid e(S, x) = \alpha_t \text{ pro } \forall x \in X_t\} \cup \mathcal{B}_t$

Computation of the nucleolus

- $LP(0), LP(1), \ldots, LP(2^n - 2)$
  - From the proof of non-emptyness theorem: $X_t \neq \emptyset$
  - at most $2^n - 1$ LPs
    - if X_t contains single vector $\implies$ it is the nucleolus
- there are more efficient algorithms for special classes of games
  - a lot of research in this nowadays

Nucleolus

Nucleolus is a single-point solution concept, which is defined for every game with **non-empty imputation set**. Moreover, if the core of this game is non-empty, the nucleolus is contained by it. In general, it is possible to compute the nucleolus by a system of up to $2^n - 1$ linear programs. Therefore, one of the most important open problem in cooperative game theory is to find efficient algorithms when restricted to special classes of games.