

COOPERATIVE GAME THEORY

MARTIN ČERNÝ

KAM.MFF.CUNI.CZ/~CERNY
CERNY@KAM.MFF.CUNI.CZ

MARCH 2, 2023

NUCLEOLUS

- $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid x(S) \geq v(S) \text{ pro } S \subseteq N\}$... core of a game
- How to choose $x \in \mathcal{C}(v)$?

- $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid x(S) \geq v(S) \text{ pro } S \subseteq N\}$... core of a game
- How to choose $x \in \mathcal{C}(v)$?
- $x(S) \geq v(S)$

- $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid x(S) \geq v(S) \text{ pro } S \subseteq N\}$... core of a game
- How to choose $x \in \mathcal{C}(v)$?

- $x(S) \geq v(S)$
- $x(S) + e = v(S)$
 - ▶ $e \leq 0$
 - ▶ size of e represents *dissatisfaction* of players over x

- $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid x(S) \geq v(S) \text{ pro } S \subseteq N\}$... core of a game
- How to choose $x \in \mathcal{C}(v)$?
 - $x(S) \geq v(S)$
 - $x(S) + e = v(S)$
 - ▶ $e \leq 0$
 - ▶ size of e represents *dissatisfaction* of players over x
 - $e(S, x) := v(S) - x(S)$... **excess** of coalition S over x
 - ▶ $\min e(S, x) \implies \max x(S)$

- $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid x(S) \geq v(S) \text{ pro } S \subseteq N\}$... core of a game
- How to choose $x \in \mathcal{C}(v)$?
 - $x(S) \geq v(S)$
 - $x(S) + e = v(S)$
 - ▶ $e \leq 0$
 - ▶ size of e represents *dissatisfaction* of players over x
 - $e(S, x) := v(S) - x(S)$... **excess** of coalition S over x
 - ▶ $\min e(S, x) \implies \max x(S)$
 - $\mathcal{C}(v) = \{x \in \mathcal{I}(v) \mid e(S, x) \leq 0 \text{ pro } S \subseteq N\}$

- $e(S, x) := v(S) - x(S)$... **excess** of coalition S over x

- $e(S, x) := v(S) - x(S)$... **excess** of coalition S over x
- $\theta(x) = (e(S_1, x), e(S_2, x), \dots, e(S_{2^n-1}, x))$... vector of excesses
 - ▶ $e(S_i, x) \geq e(S_j, x)$ pro $i \leq j$

- $e(S, x) := v(S) - x(S)$... **excess** of coalition S over x
- $\theta(x) = (e(S_1, x), e(S_2, x), \dots, e(S_{2^n-1}, x))$... vector of excesses
 - ▶ $e(S_i, x) \geq e(S_j, x)$ pro $i \leq j$
- $\theta(x) \preceq_{lex} \theta(y)$... lexicographic ordering
 1. exists $k: \theta_k(x) < \theta_k(y)$
 2. for $\ell < k: \theta_\ell(x) = \theta_\ell(y)$

- $e(S, x) := v(S) - x(S)$... **excess** of coalition S over x
- $\theta(x) = (e(S_1, x), e(S_2, x), \dots, e(S_{2^n-1}, x))$... vector of excesses
 - ▶ $e(S_i, x) \geq e(S_j, x)$ pro $i \leq j$
- $\theta(x) \preceq_{lex} \theta(y)$... lexicographic ordering

Nucleolus

For a cooperative game (N, v) , the **nucleolus** $\eta(v)$

$$\eta(v) := \{x \in \mathcal{I}(v) \mid \theta(x) \preceq_{lex} \theta(y) \text{ for } y \in \mathcal{I}(v)\}.$$

Non-emptiness of the nucleolus

For a cooperative game (N, v) , it holds

$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

NUCLEOLUS

Non-emptiness of the nucleolus

For a cooperative game (N, v) , it holds

$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

Proof: Iterative usage of the Extreme value theorem

Extreme value theorem

A **continuous function** over **compact set** \mathcal{M} attains its **minimum**.
Moreover, the set of all minima forms a **compact set** \mathcal{M} .

NUCLEOLUS

Non-emptiness of the nucleolus

For a cooperative game (N, v) , it holds

$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

Proof: Iterative usage of the Extreme value theorem

Extreme value theorem

A **continuous function** over **compact set** \mathcal{M} attains its **minimum**.
Moreover, the set of all minima forms a **compact set** \mathcal{M} .

NUCLEOLUS

Non-emptiness of the nucleolus

For a cooperative game (N, v) , it holds

$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

Proof: Iterative usage of the Extreme value theorem

Extreme value theorem

A **continuous function** over **compact set** \mathcal{M} attains its **minimum**.
Moreover, the set of all minima forms a **compact set** \mathcal{M} .

- $e(S, \bullet)$ is continuous for $S \subseteq N \implies \theta_t(\bullet)$ is continuous

Non-emptiness of the nucleolus

For a cooperative game (N, v) , it holds

$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

Proof: Iterative usage of the Extreme value theorem

Extreme value theorem

A **continuous function** over **compact set** \mathcal{M} attains its **minimum**.
Moreover, the set of all minima forms a **compact set** \mathcal{M} .

- $e(S, \bullet)$ is continuous for $S \subseteq N \implies \theta_t(\bullet)$ is continuous
- $X_0 := \mathcal{I}(v)$

NUCLEOLUS

Non-emptiness of the nucleolus

For a cooperative game (N, v) , it holds

$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

Proof: Iterative usage of the Extreme value theorem

Extreme value theorem

A **continuous function** over **compact set** \mathcal{M} attains its **minimum**.
Moreover, the set of all minima forms a **compact set** \mathcal{M} .

- $e(S, \bullet)$ is continuous for $S \subseteq N \implies \theta_t(\bullet)$ is continuous
- $X_0 := \mathcal{I}(v)$
- $X_t := \{x \in X_{t-1} \mid \theta_t(x) \leq \theta_t(y) \text{ for } y \in X_{t-1}\}$

NUCLEOLUS

Non-emptiness of the nucleolus

For a cooperative game (N, v) , it holds

$$\mathcal{I}(v) \neq \emptyset \implies \eta(v) \neq \emptyset.$$

Proof: Iterative usage of the Extreme value theorem

Extreme value theorem

A **continuous function** over **compact set** \mathcal{M} attains its **minimum**.
Moreover, the set of all minima forms a **compact set** \mathcal{M} .

- $e(S, \bullet)$ is continuous for $S \subseteq N \implies \theta_t(\bullet)$ is continuous
- $X_0 := \mathcal{I}(v)$
- $X_t := \{x \in X_{t-1} \mid \theta_t(x) \leq \theta_t(y) \text{ for } y \in X_{t-1}\}$
 - ▶ $t = 1, \dots, 2^n - 1$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

- $0 \leq \alpha \leq 1$

- $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 \leq \alpha \leq 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \succeq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 \leq \alpha \leq 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) \succeq_{lex} \theta(x) = \theta(y)$

2. $\theta(\alpha x + (1 - \alpha)y) \preceq_{lex} \theta(x) = \theta(y)$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 \leq \alpha \leq 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \succeq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \preceq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

3. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 \leq \alpha \leq 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \succeq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \preceq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

3. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

4. $\mathbf{x} = \mathbf{y}$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 \leq \alpha \leq 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) \succeq_{lex} \theta(x) = \theta(y)$

2. $\theta(\alpha x + (1 - \alpha)y) \preceq_{lex} \theta(x) = \theta(y)$

3. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

4. $x = y$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 \leq \alpha \leq 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \succeq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

▶ from definition of $\eta(\mathbf{v})$ and $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v})$

2. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \preceq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

3. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

4. $\mathbf{x} = \mathbf{y}$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 \leq \alpha \leq 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) \succeq_{lex} \theta(x) = \theta(y)$

2. $\theta(\alpha x + (1 - \alpha)y) \preceq_{lex} \theta(x) = \theta(y)$

3. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

4. $x = y$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 \leq \alpha \leq 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) \succeq_{lex} \theta(x) = \theta(y)$

2. $\theta(\alpha x + (1 - \alpha)y) \preceq_{lex} \theta(x) = \theta(y)$

▶ $\theta(\alpha x + (1 - \alpha)y) \preceq_{lex} \alpha\theta(x) + (1 - \alpha)\theta(y)$

▶ home exercise

3. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

4. $x = y$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 \leq \alpha \leq 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \succeq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \preceq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

3. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

4. $\mathbf{x} = \mathbf{y}$

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 \leq \alpha \leq 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \succeq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \preceq_{lex} \theta(\mathbf{x}) = \theta(\mathbf{y})$

3. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

▶ \succeq_{lex} is antisymmetric ordering

4. $\mathbf{x} = \mathbf{y}$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 < \alpha < 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

2. $x = y$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 < \alpha < 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

2. $x = y$

▶ $\theta(\alpha x + (1 - \alpha)y) = \alpha a + (1 - \alpha)b$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 < \alpha < 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

2. $x = y$

▶ $\theta(\alpha x + (1 - \alpha)y) = \alpha a + (1 - \alpha)b$

■ $S_1, S_2, \dots, S_{2^n - 1}$... pořadí koalic v $\theta(\alpha x + (1 - \alpha)y)$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 < \alpha < 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

2. $x = y$

▶ $\theta(\alpha x + (1 - \alpha)y) = \alpha a + (1 - \alpha)b$

■ $S_1, S_2, \dots, S_{2^n-1}$... pořadí koalic v $\theta(\alpha x + (1 - \alpha)y)$

■ $a = (e(S_1, x), e(S_2, x), \dots, e(S_{2^n-1}, x))$

■ $b = (e(S_1, y), e(S_2, y), \dots, e(S_{2^n-1}, y))$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(x) = \theta(y) \implies x = y$ "

■ $0 < \alpha < 1$

■ $x, y \in \eta(v) \implies \theta(x) = \theta(y)$

1. $\theta(\alpha x + (1 - \alpha)y) = \theta(x) = \theta(y)$

2. $x = y$

▶ $\theta(\alpha x + (1 - \alpha)y) = \alpha a + (1 - \alpha)b$

■ $S_1, S_2, \dots, S_{2^n-1}$... pořadí koalic v $\theta(\alpha x + (1 - \alpha)y)$

■ $a = (e(S_1, x), e(S_2, x), \dots, e(S_{2^n-1}, x))$

■ $b = (e(S_1, y), e(S_2, y), \dots, e(S_{2^n-1}, y))$

▶ $\theta(x) = a, \theta(y) = b$

■ $\alpha = 1, \alpha = 0$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 < \alpha < 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\mathbf{x} = \mathbf{y}$

▶ $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \alpha\mathbf{a} + (1 - \alpha)\mathbf{b}$

■ $S_1, S_2, \dots, S_{2^n-1}$... pořadí koalic v $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y})$

■ $\mathbf{a} = (e(S_1, \mathbf{x}), e(S_2, \mathbf{x}), \dots, e(S_{2^n-1}, \mathbf{x}))$

■ $\mathbf{b} = (e(S_1, \mathbf{y}), e(S_2, \mathbf{y}), \dots, e(S_{2^n-1}, \mathbf{y}))$

▶ $\theta(\mathbf{x}) = \mathbf{a}, \theta(\mathbf{y}) = \mathbf{b}$

■ $\alpha = 1, \alpha = 0$

▶ $e(S_i, \mathbf{x}) = e(S_i, \mathbf{y})$ pro $i = 1, \dots, 2^n - 1$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 < \alpha < 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\mathbf{x} = \mathbf{y}$

▶ $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \alpha\mathbf{a} + (1 - \alpha)\mathbf{b}$

■ $S_1, S_2, \dots, S_{2^n-1}$... pořadí koalic v $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y})$

■ $\mathbf{a} = (e(S_1, \mathbf{x}), e(S_2, \mathbf{x}), \dots, e(S_{2^n-1}, \mathbf{x}))$

■ $\mathbf{b} = (e(S_1, \mathbf{y}), e(S_2, \mathbf{y}), \dots, e(S_{2^n-1}, \mathbf{y}))$

▶ $\theta(\mathbf{x}) = \mathbf{a}, \theta(\mathbf{y}) = \mathbf{b}$

■ $\alpha = 1, \alpha = 0$

▶ $e(S_i, \mathbf{x}) = e(S_i, \mathbf{y})$ pro $i = 1, \dots, 2^n - 1$

■ $e(\{i\}, \mathbf{x}) = e(\{i\}, \mathbf{y})$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 < \alpha < 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\mathbf{x} = \mathbf{y}$

▶ $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \alpha\mathbf{a} + (1 - \alpha)\mathbf{b}$

■ $S_1, S_2, \dots, S_{2^n-1}$... pořadí koalic v $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y})$

■ $\mathbf{a} = (e(S_1, \mathbf{x}), e(S_2, \mathbf{x}), \dots, e(S_{2^n-1}, \mathbf{x}))$

■ $\mathbf{b} = (e(S_1, \mathbf{y}), e(S_2, \mathbf{y}), \dots, e(S_{2^n-1}, \mathbf{y}))$

▶ $\theta(\mathbf{x}) = \mathbf{a}, \theta(\mathbf{y}) = \mathbf{b}$

■ $\alpha = 1, \alpha = 0$

▶ $e(S_i, \mathbf{x}) = e(S_i, \mathbf{y})$ pro $i = 1, \dots, 2^n - 1$

■ $e(\{i\}, \mathbf{x}) = e(\{i\}, \mathbf{y})$

■ $v(i) - x_i = v(i) - y_i$

NUCLEOLUS

Uniqueness of the nucleolus

Nucleolus is a single-point solution concept.

Proof: " $\theta(\mathbf{x}) = \theta(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$ "

■ $0 < \alpha < 1$

■ $\mathbf{x}, \mathbf{y} \in \eta(\mathbf{v}) \implies \theta(\mathbf{x}) = \theta(\mathbf{y})$

1. $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \theta(\mathbf{x}) = \theta(\mathbf{y})$

2. $\mathbf{x} = \mathbf{y}$

▶ $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = \alpha\mathbf{a} + (1 - \alpha)\mathbf{b}$

■ $S_1, S_2, \dots, S_{2^n-1}$... pořadí koalic v $\theta(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y})$

■ $\mathbf{a} = (e(S_1, \mathbf{x}), e(S_2, \mathbf{x}), \dots, e(S_{2^n-1}, \mathbf{x}))$

■ $\mathbf{b} = (e(S_1, \mathbf{y}), e(S_2, \mathbf{y}), \dots, e(S_{2^n-1}, \mathbf{y}))$

▶ $\theta(\mathbf{x}) = \mathbf{a}, \theta(\mathbf{y}) = \mathbf{b}$

■ $\alpha = 1, \alpha = 0$

▶ $e(S_i, \mathbf{x}) = e(S_i, \mathbf{y})$ pro $i = 1, \dots, 2^n - 1$

■ $e(\{i\}, \mathbf{x}) = e(\{i\}, \mathbf{y})$

■ $v(i) - x_i = v(i) - y_i$

■ $x_i = y_i$

NUCLEOLUS AS A CORE SELECTOR

Nucleolus is a core selector

For a cooperative game (N, v) , if it holds $\mathcal{C}(v) \neq \emptyset$ then $\eta(v) \subseteq \mathcal{C}(v)$.

Proof: $x \notin \mathcal{C}(v)$ in contradiction with lexicographical minimality

Nucleolus is a core selector

For a cooperative game (N, v) , if it holds $\mathcal{C}(v) \neq \emptyset$ then $\eta(v) \subseteq \mathcal{C}(v)$.

Proof: $x \notin \mathcal{C}(v)$ in contradiction with lexicographical minimality

$$\blacksquare x \in \eta(v) \setminus \mathcal{C}(v) \implies \exists S \subseteq N : e(S, x) > 0$$

Nucleolus is a core selector

For a cooperative game (N, v) , if it holds $\mathcal{C}(v) \neq \emptyset$ then $\eta(v) \subseteq \mathcal{C}(v)$.

Proof: $x \notin \mathcal{C}(v)$ in contradiction with lexicographical minimality

- $x \in \eta(v) \setminus \mathcal{C}(v) \implies \exists S \subseteq N : e(S, x) > 0$
- $\text{pro } y \in \mathcal{C}(v) : e(S, y) \leq 0$

Nucleolus is a core selector

For a cooperative game (N, v) , if it holds $\mathcal{C}(v) \neq \emptyset$ then $\eta(v) \subseteq \mathcal{C}(v)$.

Proof: $x \notin \mathcal{C}(v)$ in contradiction with lexicographical minimality

- $x \in \eta(v) \setminus \mathcal{C}(v) \implies \exists S \subseteq N : e(S, x) > 0$
- $\text{pro } y \in \mathcal{C}(v) : e(S, y) \leq 0$

Nucleolus is a core selector

For a cooperative game (N, v) , if it holds $\mathcal{C}(v) \neq \emptyset$ then $\eta(v) \subseteq \mathcal{C}(v)$.

Proof: $x \notin \mathcal{C}(v)$ in contradiction with lexicographical minimality

- $x \in \eta(v) \setminus \mathcal{C}(v) \implies \exists S \subseteq N : e(S, x) > 0$
- $\text{pro } y \in \mathcal{C}(v) : e(S, y) \leq 0$
- $\theta(x) \succ_{\text{lex}} \theta(y)$
 - ▶ $\theta_1(x) > \theta_1(y)$

COMPUTATION OF THE NUCLEOLUS

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness
 - ▶ $X_0 := \mathcal{I}(v)$

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness

- ▶ $X_0 := \mathcal{I}(v)$

- ▶ $X_t := \{x \in X_{t-1} \mid \theta_t(y) \geq \theta_t(x) \text{ pro } y \in X_{t-1}\}$

- $t = 1, \dots, 2^n - 1$

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness

- ▶ $X_0 := \mathcal{I}(v)$

- ▶ $X_t := \{x \in X_{t-1} \mid \theta_t(y) \geq \theta_t(x) \text{ pro } y \in X_{t-1}\}$

- $t = 1, \dots, 2^n - 1$

- $LP(o) = \begin{cases} \min_{x \in X_0, \alpha_0 \in \mathbb{R}} & \alpha_0 \\ \text{subject to} & \alpha_0 \geq x(S) - v(S) \\ & \forall S \in 2^N \setminus \{\emptyset\} \end{cases}$

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness

- ▶ $X_0 := \mathcal{I}(v)$

- ▶ $X_t := \{x \in X_{t-1} \mid \theta_t(y) \geq \theta_t(x) \text{ pro } y \in X_{t-1}\}$

- $t = 1, \dots, 2^n - 1$

- $LP(o) = \begin{cases} \min_{x \in X_0, \alpha_0 \in \mathbb{R}} & \alpha_0 \\ \text{subject to} & \alpha_0 \geq x(S) - v(S) \\ & \forall S \in 2^N \setminus \{\emptyset\} \end{cases}$

- ▶ X_1 ... set of optimal solutions $LP(o)$

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness

- ▶ $X_0 := \mathcal{I}(v)$
- ▶ $X_t := \{x \in X_{t-1} \mid \theta_t(y) \geq \theta_t(x) \text{ pro } y \in X_{t-1}\}$
 - $t = 1, \dots, 2^n - 1$

- $LP(o) = \begin{cases} \min_{x \in X_0, \alpha_0 \in \mathbb{R}} & \alpha_0 \\ \text{subject to} & \alpha_0 \geq x(S) - v(S) \\ & \forall S \in 2^N \setminus \{\emptyset\} \end{cases}$

- ▶ X_1 ... set of optimal solutions $LP(o)$
- ▶ $B_1 = \{S \subseteq 2^N \setminus \{\emptyset\} \mid e(S, x) = \alpha_0 \text{ pro } \forall x \in X_1\}$

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness

- ▶ $X_0 := \mathcal{I}(v)$
- ▶ $X_t := \{x \in X_{t-1} \mid \theta_t(y) \geq \theta_t(x) \text{ pro } y \in X_{t-1}\}$
 - $t = 1, \dots, 2^n - 1$

- $LP(0) = \begin{cases} \min_{x \in X_0, \alpha_0 \in \mathbb{R}} & \alpha_0 \\ \text{subject to} & \alpha_0 \geq x(S) - v(S) \\ & \forall S \in 2^N \setminus \{\emptyset\} \end{cases}$

- ▶ X_1 ... set of optimal solutions $LP(0)$
- ▶ $B_1 = \{S \subseteq 2^N \setminus \{\emptyset\} \mid e(S, x) = \alpha_0 \text{ pro } \forall x \in X_1\}$

- $LP(t) = \begin{cases} \min_{x \in X_t, \alpha_t \in \mathbb{R}} & \alpha_t \\ \text{subject to} & \alpha_t \geq x(S) - v(S) \\ & \forall \emptyset \neq S \in 2^N \setminus B_t \end{cases}$

COMPUTATION OF THE NUCLEOLUS

- We can use the proof of non-emptiness

- ▶ $X_0 := \mathcal{I}(v)$
- ▶ $X_t := \{x \in X_{t-1} \mid \theta_t(y) \geq \theta_t(x) \text{ pro } y \in X_{t-1}\}$
 - $t = 1, \dots, 2^n - 1$

- $LP(0) = \begin{cases} \min_{x \in X_0, \alpha_0 \in \mathbb{R}} & \alpha_0 \\ \text{subject to} & \alpha_0 \geq x(S) - v(S) \\ & \forall S \in 2^N \setminus \{\emptyset\} \end{cases}$

- ▶ X_1 ... set of optimal solutions $LP(0)$
- ▶ $B_1 = \{S \subseteq 2^N \setminus \{\emptyset\} \mid e(S, x) = \alpha_0 \text{ pro } \forall x \in X_1\}$

- $LP(t) = \begin{cases} \min_{x \in X_t, \alpha_t \in \mathbb{R}} & \alpha_t \\ \text{subject to} & \alpha_t \geq x(S) - v(S) \\ & \forall \emptyset \neq S \in 2^N \setminus B_t \end{cases}$

- ▶ X_{t+1} ... set of optimal solutions $LP(t)$
- ▶ $B_{t+1} = \{S \in 2^N \setminus (B_t \cup \{\emptyset\}) \mid e(S, x) = \alpha_t \text{ pro } \forall x \in X_t\} \cup B_t$

- $LP(0), LP(1), \dots, LP(2^n - 2)$

- $LP(0), LP(1), \dots, LP(2^n - 2)$
 - ▶ From the proof of non-emptiness theorem: $X_t \neq \emptyset$

COMPUTATION OF THE NUCLEOLUS

- $LP(0), LP(1), \dots, LP(2^n - 2)$
 - ▶ From the proof of non-emptiness theorem: $X_t \neq \emptyset$
 - ▶ at most $2^n - 1$ LPs
 - if X_t contains single vector \implies it is the nucleolus

COMPUTATION OF THE NUCLEOLUS

- $LP(0), LP(1), \dots, LP(2^n - 2)$
 - ▶ From the proof of non-emptiness theorem: $X_t \neq \emptyset$
 - ▶ at most $2^n - 1$ LPs
 - if X_t contains single vector \implies it is the nucleolus
- there are more efficient algorithms for special classes of games

COMPUTATION OF THE NUCLEOLUS

- $LP(0), LP(1), \dots, LP(2^n - 2)$
 - ▶ From the proof of non-emptiness theorem: $X_t \neq \emptyset$
 - ▶ at most $2^n - 1$ LPs
 - if X_t contains single vector \implies it is the nucleolus
- there are more efficient algorithms for special classes of games
 - ▶ a lot of research in this nowadays

Nucleolus

Nucleolus is a single-point solution concept, which is defined for every game with **non-empty imputation set**. Moreover, if the core of this game is non-empty, the nucleolus is contained by it. In general, it is possible to compute the nucleolus by a system of up to $2^n - 1$ linear programs. Therefore, one of the most important open problem in cooperative game theory is to find efficient algorithms when restricted to special classes of games.