Cooperative game theory

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The core

Cooperative game

A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

- Γ^n ... set of n-person cooperative games
- $S \subseteq N$ ... coalition
- $v(S)$ ... value of coalition
  - **Payoff vector** $\mathbf{x} \in \mathbb{R}^n$
    - x_i represents payoff of player i
    - $x(S) := \sum_{i \in S} x_i$
  - Vector $\mathbf{x} \in \mathbb{R}^n$ is **efficient**, if $x(N) = v(N)$
    - Usually, we distribute $v(N)$
  - Vector $\mathbf{x} \in \mathbb{R}^n$ is **individualy rational**, if $x_i \geq v(i)$
    - players prefer x_i over $v(i)$
  - $\mathcal{I}^*(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$ ... **preimputation**
  - $\mathcal{I}(v) = \{x \in \mathcal{I}^*(v) \mid \forall i \in N : x_i \geq v(i)\}$ ... **imputation**

Idea: *Payoff distribution leads to cooperation...*

The core

For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\}\,.$$

- $v(N)$ ... value, which is distributed among players
- $x(S) > v(S) \implies$ coalition S does not leave N
  - assumption: *homo economicus*
  - would lead to (N, v_S)
  - $v(S)$ ... distributed value

Goal: *To analyse the core.*

Idea: Deviation from the actual strategy to a new strategy does not improve the outcome.

Nash equilibrium

Strategy profile $(s_1, \ldots, s_n)$ is **Nash equilibrium**, if it holds for every player i,

$$v_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq v_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$$

for every $t_i \in S_i$.

The core

For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\}.$$

Is there always a core solution?

The core

For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\} .$$

- Nash: *My equilibrium always exists!*
- Does this hold for $\mathcal{C}(v)$?

- non-esential game (N, v)
  - $v(N) < \sum_{i \in N} v(i)$
- $x \in \mathcal{C}(v)$
  - $x(N) = v(N) < \sum_{i \in N} v(i) \leq x(N)$
    - $v(i) \leq x_i$
- x does not exist $\implies (v) = \emptyset$

On emptyness of the core

There are cooperative games (N, v) with empty core.

T	$\{H\}$	$\{S\}$	$\{I\}$	$\{H,S\}$	$\{H,I\}$	$\{S,I\}$	$\{H,S,I\}$
$v(T)$	5	2	1	8	4	4	10

- $\{H, S, I\}$ ... companies
  - H ... hardware
  - S ... software
  - I ... IT support
- $v(\{H, S, I\})$ ... price of joint product
- *Is it beneficial for the companies to sell a joint product?*
  - *How to distribute the value to support cooperation?*

Examples of the core: Joint production

T	$\{H\}$	$\{S\}$	$\{I\}$	$\{H,S\}$	$\{H,I\}$	$\{S,I\}$	$\{H,S,I\}$
$v(T)$	5	2	1	8	4	4	10

- $\{H, S, I\}$ ... companies
- *Is it beneficial for the companies to sell a joint product?*
  - *How to distribute the value to support cooperation?*

Payoff vectors from the core $x \in \mathbb{R}^3$:

1. $x(N) = v(N)$
   - $x_1 + x_2 + x_3 = 10$
2. $x(T) \geq v(T), \forall T \subseteq N$
   - $x_1 \geq 5$
   - $x_2 \geq 2$
   - $x_3 \geq 1$
   - $x_1 + x_2 \geq 8$
   - $x_1 + x_3 \geq 4$
   - $x_2 + x_3 \geq 4$
   - $x_1 + x_2 + x_3 \geq 10$

Examples of the core: Joint production

T	$\{H\}$	$\{S\}$	$\{I\}$	$\{H,S\}$	$\{H,I\}$	$\{S,I\}$	$\{H,S,I\}$
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- $\{H, S, I\}$ ... companies
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    - $x_1 \geq 5$
    - $x_2 \geq 2$
    - $x_3 \geq 1$
    - $x_1 + x_2 \geq 8$
    - ~~$x_1 + x_3 \geq 4$~~
    - $x_2 + x_3 \geq 4$
    - ~~$x_1 + x_2 + x_3 \geq 10$~~

Examples of the core: Joint production

T	$\{H\}$	$\{S\}$	$\{I\}$	$\{H,S\}$	$\{H,I\}$	$\{S,I\}$	$\{H,S,I\}$
$v(T)$	5	2	1	8	4	4	10

- $\{H, S, I\}$ ... companies
- *Is it beneficial for the companies to sell a joint product?*
    - *How to distribute the value to support cooperation?*

Payoff vectors from the core $x \in \mathbb{R}^3$: $(5.\overline{6}, 2.\overline{6}, 1.\overline{6})$

- uniform distribution of the surplus

1. $x(N) = v(N)$
    - $5.\overline{6} + 2.\overline{6} + 1.\overline{6} = 10$
2. $x(T) \geq v(T), \forall T \subseteq N$
    - $5.\overline{6} \geq 5$
    - $2.\overline{6} \geq 2$
    - $1.\overline{6} \geq 1$
    - $5.\overline{6} + 2.\overline{6} \geq 8$
    - $2.\overline{6} + 1.\overline{6} \geq 4$

T	$\{H\}$	$\{S\}$	$\{I\}$	$\{H,S\}$	$\{H,I\}$	$\{S,I\}$	$\{H,S,I\}$
$v(T)$	5	2	1	8	4	4	10

- $\{H, S, I\}$ ... companies
- *Is it beneficial for the companies to sell a joint product?*
  - *How to distribute the value to support cooperation?*

Payoff vectors from the core $x \in \mathbb{R}^3$: $(5, 4, 1)$

- company S is prefered

1. $x(N) = v(N)$
   - $5 + 4 + 1 = 10$
2. $x(T) \geq v(T), \forall T \subseteq N$
   - $5 \geq 5$
   - $4 \geq 2$
   - $1 \geq 1$
   - $5 + 4 \geq 8$
   - $4 + 1 \geq 4$

T	$\{H\}$	$\{S\}$	$\{I\}$	$\{H,S\}$	$\{H,I\}$	$\{S,I\}$	$\{H,S,I\}$
$v(T)$	5	2	1	8	4	4	10

- $\{H,S,I\}$ ... companies
- *Is it beneficial for the companies to sell a joint product?*
  - *How to distribute the value to support cooperation?*

Payoff vectors from the core $x \in \mathbb{R}^3$: $(5+\alpha, 2+\beta, 1+\gamma)$

- in general

1. $x(N) = v(N)$
   - $x_1 + x_2 + x_3 = 10$
     - $\alpha + \beta + \gamma = 2$
     - $\alpha, \beta, \gamma \geq 0$
2. $x(T) \geq v(T), \forall T \subseteq N$
   - $x_1 + x_2 \geq 8$
     - $\alpha + \beta \geq 1$
   - $x_2 + x_3 \geq 4$
     - $\beta + \gamma \geq 1$

A bank robbery

- n thieves
- 2 thieves can carry one bag of gold
- $v(S) = \begin{cases} \frac{|S|}{2} & \text{if } S \text{ is even,} \\ \frac{|S|-1}{2} & \text{if } S \text{ is odd.} \end{cases}$

What does $\mathcal{C}(v)$ look like?

A bank robbery

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- 2 thieves can carry one bag of gold
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What does $\mathcal{C}(v)$ look like? $N = \{1, 2, 3, 4\}$

- $v(N) = 2$
  - $x_1 + x_2 \geq 1$
  - $x_3 + x_4 \geq 1$
  - $x_1 + x_2 = 1$
  - $x_3 + x_4 = 1$

A bank robbery

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- 2 thieves can carry one bag of gold
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What does $\mathcal{C}(v)$ look like? $N = \{1, 2, 3, 4\}$

- $v(N) = 2 = x_1 + x_2 + x_3 + x_4$
  - $x_1 + x_3 \geq 1$
  - $x_2 + x_4 \geq 1$
  - $x_1 + x_3 = 1$
  - $x_2 + x_4 = 1$

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- 2 thieves can carry one bag of gold
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A bank robbery

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What does $\mathcal{C}(v)$ look like? $N = \{1, 2, 3, 4\}$

- $v(N) = 2 = x_1 + x_2 + x_3 + x_4$
  - $x_1 + x_2 = 1$
  - $x_1 + x_4 = 1$
  - $x_1 + x_3 = 1$
  - $x_2 + x_3 = 1$
  - $x_2 + x_4 = 1$
  - $x_3 + x_4 = 1$

$$x_1 - x_3 = 0$$

$$x_1 = x_3$$

$$\forall i \neq j : x_i = x_j$$

- $\mathcal{C}(v) = \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T\}$

A bank robbery

- n thieves
- 2 thieves can carry one bag of gold
- $v(S) = \begin{cases} \frac{|S|}{2} & \text{if } S \text{ is even,} \\ \frac{|S|-1}{2} & \text{if } S \text{ is odd.} \end{cases}$

What does $\mathcal{C}(v)$ look like? $N = \{1, 2, 3\}$

- $v(N) = 1 = x_1 + x_2 + x_3$
  - $x_1 + x_2 \geq 1$
  - $x_1 + x_3 \geq 1 \implies x_2 = 0$
  - $x_2 + x_3 \geq 1 \implies x_1 = 0$
- $\mathcal{C}(v) = \emptyset$

A bank roberry

- n thieves
- 2 thieves can carry one bag of gold
- $v(S) = \begin{cases} \frac{|S|}{2} & \text{if } S \text{ is even,} \\ \frac{|S|-1}{2} & \text{if } S \text{ is odd.} \end{cases}$

What does $\mathcal{C}(v)$ look like?

1. $n > 2$ is even
   - $\mathcal{C}(v) = \{(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})^T\}$
2. n is odd
   - $\mathcal{C}(v) = \emptyset$

- later: *The Shapley value fairly distributes* $x_i = \frac{v(N)}{n}$

Shoe selling problem

- N ... $2k+1$ players
  - k have left shoe
  - $k+1$ have right shoe
- price for pair: 1000 czk
- $v(S) = \min\{S_\ell, S_p\} \cdot 1000$
  - S_ℓ ... numbers of players in S with left shoe
  - S_p ... number of players in S with right shoe
  - $v(N) = k \cdot 1000$

What does $\mathcal{C}(v)$ look like ? $N = \{\ell, r_1, r_2\}$

- $V(N) = 1000 = x_\ell + x_{r_1} + x_{r_2}$
  - $x_\ell + x_{r_1} \geq 1000$
  - $x_\ell + x_{r_1} \geq 1000$
  - $x_{r_1} \geq 0$
  - $x_{r_2} \geq 0$
  - $\implies x_\ell = 1000$
- $\mathcal{C}(v) = (1000, 0, 0)^T$

Shoe selling problem

- N ... $2k+1$ players
  - k have left shoe
  - $k+1$ have right shoe
- price for pair: 1000 czk
- $v(S) = \min\{S_\ell, S_p\} \cdot 1000$
  - S_ℓ ... numbers of players in S with left shoe
  - S_p ... number of players in S with right shoe
  - $v(N) = k \cdot 1000$

What does $\mathcal{C}(v)$ look like?

- $\mathcal{C}(v) = \{x\}$:
  - $x_i = \begin{cases} 1000 & i \in N_\ell \\ 0 & i \in N_p \end{cases}$

- $\mathcal{B} = \{(N,v) \in \Gamma^n \mid \mathcal{C}(v) \neq \emptyset\}$ ... games with non-empty core
- *When is the core non-empty?*
  - *we employ duality of linear programming:*

  1. We encode non-emptiness question as primal program (P)
  2. We determine the dual program (D)
  3. we derive the weak Bondareva-Shapley theorem from (D)

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & c^T x \\ \text{subject to} & Ax \geq b \end{cases}$ $\quad (D) = \begin{cases} \max_{y \in \mathbb{R}^m_+} & b^T y \\ \text{subject to} & A^T y = c \end{cases}$
  - $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & c^T x \\ \text{subject to} & Ax \geq b \end{cases}$
  - $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

Conditions of the core:

1. $x(N) = v(N)$
   - we assert this by optimality
2. $x(S) \geq v(S)$ pro $S \subseteq N$
   - conditions of LP

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & x(N) \\ \text{subject to} & x(S) \geq v(S) \text{ for } S \subseteq N \end{cases}$
  - $x \in \mathcal{C}(v) \implies x(N) = v(N)$
  - $\mathcal{C}(v) \neq \emptyset \iff \min_{x \in \mathbb{R}^n} x(N) = v(N)$

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & c^T x \\ \text{subject to} & Ax \geq b \end{cases}$ $\quad (D) = \begin{cases} \max_{y \in \mathbb{R}^m_+} & b^T y \\ \text{subject to} & A^T y = c \end{cases}$
  - $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

LP encoding the core:

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & x(N) \\ & x(S) \geq v(S) \text{ pro } S \subseteq N \end{cases}$
  - $b \in \mathbb{R}^{2^n - 1}$
    - $b_S = v(S)$
    - we omit $b_\emptyset$ $(0 \geq 0)$
  - $c = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$
    - $c^T x = \sum_{i \in N} c_i \cdot x_i = \sum_{i \in N} 1 \cdot x_i = \sum_{i \in N} x_i$
  - $A \in \mathbb{R}^{(2^n - 1) \times n}$

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & c^T x \\ \text{za podm.} & Ax \geq b \end{cases}$ $\quad (D) = \begin{cases} \max_{y \in \mathbb{R}^m_+} & b^T y \\ \text{subject to} & A^T y = c \end{cases}$
  - $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

LP encoding the core:

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & x(N) \\ \text{subject to} & x(S) \geq v(S) \text{ pro } S \subseteq N \end{cases}$
- $(D) = \begin{cases} \max_{y \in \mathbb{R}_+^{(2^n - 1)}} & \sum_{S \subseteq N} v(S) y_S \\ \text{subject to} & \sum_{\emptyset \neq S \subseteq N} (\mathcal{X}_S)_i y_S = 1 \text{ pro } i \in N \end{cases}$
  - $\mathcal{X}^S$ ... characteristic vector of S
    - $(\mathcal{X}_S)_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & x(N) \\ \text{subject to} & x(S) \geq v(S) \text{ pro } S \subseteq N \end{cases}$
- $(D) = \begin{cases} \max_{y \in \mathbb{R}_+^{(2^n - 1)}} & \sum_{S \subseteq N} y_S v(S) \\ \text{subject to} & \sum_{\emptyset \neq S \subseteq N} y_S \mathcal{X}_S = \mathcal{X}_N \end{cases}$

Theorem on the relation of (P) and (D)

If (P) and (D) are both feasible, the optimum of both of the programs is attained and is the same for both programs.

- (P) is feasible for $x \in \mathbb{R}^n$ large enough
- (D) is feasible e.g. for $y \in \mathbb{R}_+^{(2^n - 1)}$, $y_S := \begin{cases} 1 & S = N \\ 0 & S \neq N \end{cases}$
- $\min_{x \in \mathbb{R}^n} x(N) = \max_{y \in \mathbb{R}_+^{(2^n - 1)}} \sum_{S \subseteq N} y_S v(S)$

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & x(N) \\ \text{subject to} & x(S) \geq v(S) \text{ pro } S \subseteq N \end{cases}$
- $(D) = \begin{cases} \max_{y \in \mathbb{R}_+^{(2^n-1)}} & \sum_{S \subseteq N} y_S v(S) \\ \text{subject to} & \sum_{\emptyset \neq S \subseteq N} y_S \mathcal{X}_S = \mathcal{X}_N \end{cases}$
- $\min_{x \in \mathbb{R}^n} x(N) = \max_{y \in \mathbb{R}_+^{(2^n-1)}} \sum_{S \subseteq N} y_S v(S)$

1. $\mathcal{C}(v) \neq \emptyset$
   - $\min_{x \in \mathbb{R}^n} x(N) = v(N) = \max_{y \in \mathbb{R}_+^{(2^n-1)}} \sum_{S \subseteq N} y_S v(S)$
   - $v(N) \geq \sum_{S \subseteq N} y_S v(S)$ for every $y \in \mathbb{R}_+^{(2^n-1)}$
2. $\mathcal{C}(v) = \emptyset$
   - $\min_{x \in \mathbb{R}^n} x(N) > v(N)$
   - $v(N) < \sum_{S \subseteq N} y_S v(S)$ for every $y \in \mathbb{R}_+^{(2^n-1)}$

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & x(N) \\ \text{subject to} & x(S) \geq v(S) \text{ pro } S \subseteq N \end{cases}$
- $(D) = \begin{cases} \max_{y \in \mathbb{R}_+^{(2^n-1)}} & \sum_{S \subseteq N} y_S v(S) \\ \text{subject to} & \sum_{\emptyset \neq S \subseteq N} y_S \mathcal{X}_S = \mathcal{X}_N \end{cases}$

Weak Bondareva-Shapley theorem

Cooperative game (N, v) has **non-empty core** if and only if

$$v(N) \geq \sum_{S \subseteq N} y_S v(S) \text{ for all feasible } y \in \mathbb{R}^{(2^n-1)}.$$

- Collection $\mathcal{B} \subseteq 2^N$ is **balanced**, if there exist $\delta_S > 0$ satisfying $\sum_{S \in \mathcal{B}} \delta_S \mathcal{X}_S = \mathcal{X}_N$,
  - δ_S ... **balancing weights**

Weak Bondareva-Shapley theorem

Cooperative game (N, v) has **non-empty core** if and only if for every balanced collection $\mathcal{B}$ and every system of balancing weights $(\delta_S)_{S \in \mathcal{B}}$, it holds

$$v(N) \geq \sum_{S \in \mathcal{B}} \delta_S v(S).$$

Totally balanced games

- (N, v) is **balanced game**
  - $\mathcal{C}(v) \neq \emptyset$
- (N, v) is **totally balanced game**
  - $\mathcal{C}(v_S) \neq \emptyset$ pro $S \subseteq N$
    - (S, v_S) subgame

Generalising the core

- (N, v) ... non-balanced game ($\mathcal{C}(v) = \emptyset$)
- *We want core-like solution...*

1. **Strong ε-core** $\mathcal{C}_\varepsilon(v)$
    - *Weakening the inequalities*
    - $\mathcal{C}_\varepsilon(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S) - \varepsilon$ pro $S \subseteq N\}$
        - we choose ε minimal
2. **Core catchers**
    - *Superset of the core, based on bounds of payoff vectors*
    - a^v, b^v ... upper and lower bound
        - $b_i^v = v(N) - v(N \setminus i)$ ... maximal right
        - $a_i^v = \max_{S \subseteq N : i \in S} \{v(S) - b^v(S \setminus i)\}$ ... minimal right
    - $x \in \mathcal{C}(v) \implies a^v \leq x \leq b^v$
    - $\mathcal{CC}(v) = \{x \in \mathcal{I}^*(v) \mid a^v \leq x \leq b^v\}$ ... **core catcher**
    - $\mathcal{H}(v) = \{x \in \mathbb{R}^n \mid a^v \leq x \leq b^v\}$ ... **hypercube**

- $\mathcal{C}(v)$ is a convex set
  - since it is given by a system of linear conditions (in/equalities)
- $\mathcal{C}(v)$ je bounded
  - $\mathcal{C}(v) \subseteq \mathcal{I}(v)$
  - $\mathcal{I}(v) = \{x_\beta \in \mathbb{R}^n \mid \beta \in \mathbb{R}^n_+ \text{ a } \beta(N) = 1\}$
    - $(x_\beta)_i := v(i) + \beta_i \Delta$
    - $\Delta := v(N) - \sum_{i \in N} v(i)$
- the structure of $\mathcal{C}(v)$ is difficult to describe in general
  - for **convex games**, we have nice structure
    - later during the semester
    - similar results for *k-konvexní* games and other...

Core selections

- How to choose $x \in \mathcal{C}(v)$, if there is more than one?
  1. *nucleolus*
     - next lecture
  2. *egalitarian core*
     - maybe later during the semester
  3. ...

The core of a cooperative game

The **core** is a multi-point solution concept, which contitutes of *stable* payoff vectors, i.e. payoff vectors leading to cooperation of all players. For general cooperative game, the core might be empty. Game with non-empty core are called **balanced**. Games with non-empty core for each of its subgames are **totally balanced**. The core is sensitive to *imbalance of values*.