Cooperative game theory

Martin Černý

kam.mff.cuni.cz/∼cerny
cerny@kam.mff.cuni.cz

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The core

Cooperative game

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A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

- Γ^n ... set of n-person cooperative games
- $S \subseteq N$ ... coalition
- $v(S)$ ... value of coalition
  - **Payoff vector $\mathbf{x}$** $\in \mathbb{R}^n$
    - x_i represents payoff of player i
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  - Vector $\mathbf{x} \in \mathbb{R}^n$ is **efficient**, if $x(N) = v(N)$
    - Usually, we distribute $v(N)$
  - Vector $\mathbf{x} \in \mathbb{R}^n$ is **individualy rational**, if $x_i \geq v(i)$
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    - players prefer x_i over $v(i)$
  - $\mathcal{I}^*(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$ ... **preimputation**
  - $\mathcal{I}(v) = \{x \in \mathcal{I}^*(v) \mid \forall i \in N : x_i \geq v(i)\}$ ... **imputation**

Goal of the day: The core

Idea: *Payoff distribution leads to cooperation...*

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Goal: *To analyse the core.*

Idea: Deviation from the actual strategy to a new strategy does not improve the outcome.

Nash equilibrium

Strategy profile $(s_1, \ldots, s_n)$ is **Nash equilibrium**, if it holds for every player i,

$$v_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq v_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$$

for every $t_i \in S_i$.

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    - $v(i) \leq x_i$
- x does not exist $\implies (v) = \emptyset$

On emptyness of the core

There are cooperative games (N, v) with empty core.

Examples of the core: Joint production

T	$\{H\}$	$\{S\}$	$\{I\}$	$\{H, S\}$	$\{H, I\}$	$\{S, I\}$	$\{H, S, I\}$
$v(T)$	5	2	1	8	4	4	10

- $\{H, S, I\}$ ... companies
  - H ... hardware
  - S ... software
  - I ... IT support

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- *Is it beneficial for the companies to sell a joint product?*
  - *How to distribute the value to support cooperation?*

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Payoff vectors from the core $x \in \mathbb{R}^3$:

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   - $x_1 \geq 5$
   - $x_2 \geq 2$
   - $x_3 \geq 1$
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Payoff vectors from the core $x \in \mathbb{R}^3$: $(5.\overline{6}, 2.\overline{6}, 1.\overline{6})$

- uniform distribution of the surplus

1. $x(N) = v(N)$
   - $5.\overline{6} + 2.\overline{6} + 1.\overline{6} = 10$
2. $x(T) \geq v(T), \forall T \subseteq N$
   - $5.\overline{6} \geq 5$
   - $2.\overline{6} \geq 2$
   - $1.\overline{6} \geq 1$
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Payoff vectors from the core $x \in \mathbb{R}^3$: $(5,4,1)$

- company S is prefered

1. $x(N) = v(N)$
   - $5 + 4 + 1 = 10$
2. $x(T) \geq v(T), \forall T \subseteq N$
   - $5 \geq 5$
   - $4 \geq 2$
   - $1 \geq 1$
   - $5 + 4 \geq 8$
   - $4 + 1 \geq 4$

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Payoff vectors from the core $x \in \mathbb{R}^3$: $(5+\alpha, 2+\beta, 1+\gamma)$

- in general

1. $x(N) = v(N)$
   - $x_1 + x_2 + x_3 = 10$
     - $\alpha + \beta + \gamma = 2$
     - $\alpha, \beta, \gamma \geq 0$
2. $x(T) \geq v(T), \forall T \subseteq N$
   - $x_1 + x_2 \geq 8$
     - $\alpha + \beta \geq 1$
   - $x_2 + x_3 \geq 4$
     - $\beta + \gamma \geq 1$

Core disadvantages: Sensitivity to imbalance (1)

A bank robbery

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What does $\mathcal{C}(v)$ look like? $N = \{1, 2, 3, 4\}$

- $v(N) = 2$

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- $v(N) = 2 = x_1 + x_2 + x_3 + x_4$
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$$x_1 = x_2$$

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$$x_i = x_j$$

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What does $\mathcal{C}(v)$ look like? $N = \{1, 2, 3, 4\}$

- $v(N) = 2 = x_1 + x_2 + x_3 + x_4$
  - $x_1 + x_2 = 1$
  - $x_1 + x_4 = 1$
  - $x_1 + x_3 = 1$
  - $x_2 + x_3 = 1$
  - $x_2 + x_4 = 1$
  - $x_3 + x_4 = 1$

$$x_1 - x_3 = 0$$

$$x_1 = x_2$$

$$x_i = x_j$$

- $\mathcal{C}(v) = \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T\}$

A bank robbery

- n thieves
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  - $x_1 + x_2 \geq 1$

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  - $x_2 + x_3 \geq 1 \implies x_3 = 0$
- $(v) = \emptyset$

A bank roberry

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What does $\mathcal{C}(v)$ look like?

1. $n > 2$ is even
    - $\mathcal{C}(v) = \{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})^T\}$

Core disadvantages: Sensitivity to imbalance (1)

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- later: *The Shapley value fairly distributes* $x_i = \frac{v(N)}{n}$

Core disadvantages: Sensitivity to imbalance (2)

Shoe selling problem

Shoe selling problem

- N ... $2k + 1$ players
  - k have left shoe
  - k have right shoe

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- $V(N) = 1000 = x_\ell + x_{r_1} + x_{r_2}$

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  - $\implies x_\ell = 1000$
- $\mathcal{C}(v) = (1000, 0, 0)^T$

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What does $\mathcal{C}(v)$ look like?

- $\mathcal{C}(v) = \{x\}$:
  - $x_i = \begin{cases} 1000 & i \in N_\ell \\ 0 & i \in N_p \end{cases}$

When is the core non-empty?

- $\mathcal{B} = \{(N, v) \in \Gamma^n \mid \mathcal{C}(v) \neq \emptyset\}$ ... games with non-empty core
- *When is the core non-empty?*

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  - *we employ duality of linear programming:*

- $(P) = \begin{cases} \min_{x \in \mathbb{R}^n} & c^T x \\ \text{subject to} & Ax \geq b \end{cases}$ $\qquad (D) = \begin{cases} \max_{y \in \mathbb{R}^m_+} & b^T y \\ \text{subject to} & A^T y = c \end{cases}$
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  3. we derive the weak Bondareva-Shapley theorem from (D)

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Conditions of the core:

1. $x(N) = v(N)$
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  - $\mathcal{X}^S$ ... characteristic vector of S
    - $(\mathcal{X}_S)_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$

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Theorem on the relation of (P) and (D)

If (P) and (D) are both feasible, the optimum of both of the programs is attained and is the same for both programs.

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Cooperative game (N, v) has **non-empty core** if and only if

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Totally balanced games

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  - $\mathcal{C}(v_S) \neq \emptyset$ pro $S \subseteq N$
    - (S, v_S) subgame

Generalising the core

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   - $\mathcal{H}(v) = \{x \in \mathbb{R}^n \mid a^v \leq x \leq b^v\}$ ... **hypercube**

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- the structure of $\mathcal{C}(v)$ is difficult to describe in general
  - for **convex games**, we have nice structure
    - later during the semester
    - similar results for *k-konvexní* games and other...

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Core selections

- How to choose $x \in \mathcal{C}(v)$, if there is more than one?
  1. *nucleolus*
     - next lecture
  2. *egalitarian core*
     - maybe later during the semester
  3. ...

The core of a cooperative game

The **core** is a multi-point solution concept, which contitutes of *stable* payoff vectors, i.e. payoff vectors leading to cooperation of all players. For general cooperative game, the core might be empty. Game with non-empty core are called **balanced**. Games with non-empty core for each of its subgames are **totally balanced**. The core is sensitive to *imbalance of values*.