

COOPERATIVE GAME THEORY

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SOCIAL CHOICE AND IMPOSSIBILITY THEOREMS

EXAMPLE OF SOCIAL CHOICE

Agent	a_1	a_2	a_3	a_4	a_5
1	5	1	3	2	4
2	1	2	3	4	5
3	3	4	5	2	1

■ $\{a_1, \dots, a_5\}$... alternative

■ Preferences

- ▶ **Agent 1:** $a_2 \succ_1 a_4 \succ_1 a_3 \succ_1 a_5 \succ_1 a_1$
- ▶ **Agent 2:** $a_1 \succ_2 a_2 \succ_2 a_3 \succ_2 a_4 \succ_2 a_5$
- ▶ **Agent 3:** $a_5 \succ_3 a_4 \succ_3 a_1 \succ_2 a_5 \succ_3 a_3$

Goal: What is the best **social choice** among the alternatives?

WHAT IS THE BEST **SOCIAL CHOICE** AMONG THE ALTERNATIVES?

Agent	a_1	a_2	a_3	a_4	a_5
1	5	1	3	2	4
2	1	2	3	4	5
3	3	4	5	2	1
Choice?					

■ Social preference

▶ $a_2 \prec a_4 \prec a_1 \prec a_5 \prec a_3$

■ a_2 ... social choice

■ Disadvantage of this approach?

▶ ranking of two alternatives **may** depend **not** solely on individual preferences between the alternatives

WHAT IS THE BEST **SOCIAL CHOICE** AMONG THE ALTERNATIVES? **NOT** BORDA SCORES

Agent	a_1	a_2	a_3	a_4	a_5
1	5	1	3	2	4
2	1	2	3	4	5
3	3	4	5	2	1
Σ	9	7	11	8	10

■ Social preference: $a_2 \succ a_4 \succ a_1 \succ a_5 \succ a_3$

▶ $a_4 \succ_1 a_1$

▶ $a_4 \succ_2 a_1$

▶ $a_4 \succ_3 a_1$

■ Social preference: $a_2 \succ a_5 \succ a_1 \succ a_4 \succ a_3$

▶ $a_4 \succ_1 a_1$

▶ $a_4 \succ_2 a_1$

▶ $a_4 \succ_3 a_1$

■ individual preference of a_1 and a_4 did not change

■ social preference **did!**

WHAT IS THE BEST **SOCIAL CHOICE** AMONG THE ALTERNATIVES? **NOT** BORDA SCORES

- *Can we avoid this behaviour?*
- *Arrow: Only by selecting a dictator!*

FORMAL MODEL

- a_1, \dots, a_k ... alternatives
- $\succsim \subseteq A \times A$... preference
 1. (reflexive): $a \succsim a$ for all $a \in A$
 2. (complete): $a \succsim b$ or $b \succsim a$ for all $a, b \in A$
 3. (transitive) $a \succsim b, b \succsim c \implies a \succsim c$ for all $a, b, c \in A$
- ▶ $\prec \subseteq A \times A$... strict preference
 1. \prec ... preference
 2. (antisymmetric): $a \prec b, b \prec a \implies a = b$
- \mathcal{P}_{\succsim} ... set of preferences on A
- \mathcal{P}_{\prec} ... set of strict preferences on A
 - ▶ $(\prec_1, \dots, \prec_n) \in \mathcal{P}_{\prec}^n$... strict preference profile
- $F: \mathcal{P}_{\prec}^n \rightarrow \mathcal{P}_{\succsim}$... social welfare function
 - ▶ e.g. $F(\prec_1, \dots, \prec_n) = \succsim$
- $f: \mathcal{P}_{\prec}^n \rightarrow A$... social choice function
 - ▶ e.g. $f(\prec_1, \dots, \prec_n) = a$

SOCIAL WELFARE FUNCTION

Social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is

1. (pareto efficient) $a \succ_i b$ for all $i \in N \implies a \succ b$

▶ *Preferred by all, preferred by society*

2. (independent of irrelevant alternatives)

$\forall i \in N : a \succ_i b \iff a \succ_i^* b \implies a \preceq b \iff a \preceq^* b$

▶ *Preference of alternatives determines their social preference*

3. (Dictatorial) $\exists i \in N : F(\succ_1, \dots, \succ_n) = \succ_i$

▶ *There is a dictator!*

ARROW'S THEOREM

Arrow's theorem

A Pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof: Start with $(\succsim_1, \succsim_2, \dots, \succsim_n)$ s.t. $a \succ_i b$ for $i \in N$

■ Pareto optimality $\implies a \succ b$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
a	a	...	a	a	a	...	a	a	a
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
b	b	...	b	b	b	...	b	b	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move b up in \succsim_1

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
a	a	...	a	a	a	...	a	a	a
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
b	•	...	•	•	•	...	•	•	
•	b	...	b	b	b	...	b	b	

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Proof:

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\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
a	a	...	a	a	a	...	a	a	a
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
b	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	b	
•	b	...	b	b	b	...	b	•	

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Proof:

- move b up in \succsim_1

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
a	a	...	a	a	a	...	a	a	a
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	b	
•	b	...	b	b	b	...	b	•	

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Proof:

- move b up in \succsim_1

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
a	a	...	a	a	a	...	a	a	a
•	•	...	•	•	•	...	•	•	
b	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	b	
•	b	...	b	b	b	...	b	•	

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Proof:

- move b up in \succsim_1

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
a	a	...	a	a	a	...	a	a	a
b	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	b	
•	•	...	•	•	•	...	•	•	
•	b	...	b	b	•	...	b	•	

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Proof:

- move b up in \succsim_1

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	a	...	a	a	a	...	a	a	a
a	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	b	
•	•	...	•	•	•	...	•	•	
•	b	...	b	b	•	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move b up in \succsim_2

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	a	...	a	a	a	...	a	a	a
a	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	b	...	•	•	•	...	•	•	
•	•	...	b	b	•	...	b	•	

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A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move b up in \succsim_2

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	a	a	a	...	a	a	a
a	a	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
•	•	...	b	b	•	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move b up in \succsim_j until it holds $a \succ b$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	a	a	a	...	a	a	a
a	a	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
•	•	...	b	b	•	...	b	•	

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A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move b up in \succsim_j until it holds $a \succ b$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	a	a	...	a	a	a
a	a	...	a	b	•	...	•	•	
•	•	...	•	•	•	...	•	b	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

■ move b up in $\succsim_k \implies a \not\prec b$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	b	a	...	a	b	b
a	a	...	a	a	•	...	•	a	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move a to the bottom in \succsim_1

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	b	a	...	a	b	b
•	a	...	a	a	•	...	•	a	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
a	•	...	•	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move a to the bottom in \succsim_2

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	b	a	...	a	b	b
•	•	...	a	a	•	...	•	•	
•	•	...	•	•	•	...	•	a	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
a	a	...	•	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move a to the bottom in \succsim_{k-1}

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	b	a	...	a	b	b
•	•	...	•	a	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
a	a	...	a	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move a just above in $b \succ_{k+1}$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	b	•	...	a	b	b
•	•	...	•	a	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	a	...	•	•	
a	a	...	a	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- move a just above b in $\succsim_{k+1}, \dots, \succsim_n$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	b	•	...	•	b	b
•	•	...	•	a	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	a	
•	•	...	•	•	a	...	a	•	
a	a	...	a	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

■ change $b \prec_k a$ to $a \prec_k b$

\prec_1	\prec_2	...	\prec_{k-1}	\prec_k	\prec_{k+1}	...	\prec_n	F	f
b	b	...	b	a	•	...	•	?	?
•	•	...	•	b	•	...	•	?	
•	•	...	•	•	•	...	•	?	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	?	
•	•	...	•	•	a	...	a	?	
a	a	...	a	•	b	...	b	?	

ARROW'S THEOREM - OPTION 1

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- IIA: Only order of a, b can change $\implies b$ (almost) at the top

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	a	•	...	•	b	b
•	•	...	•	b	•	...	•	a	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	a	...	a	•	
a	a	...	a	•	b	...	b	•	

ARROW'S THEOREM - OPTION 2

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- IIA: Only order of a, b can change $\implies b$ (almost) at the top

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	a	•	...	•	a	a
•	•	...	•	b	•	...	•	b	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	a	...	a	•	
a	a	...	a	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- recall previous table and move a to the bottom

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	a	a	...	a	a	a
a	a	...	a	b	•	...	•	•	
•	•	...	•	•	•	...	•	b	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	b	...	b	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

■ preferences between a, b do **not** change $\implies a \succsim b$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	a	•	...	•	?	?
•	•	...	•	b	•	...	•	?	
•	•	...	•	•	•	...	•	?	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	?	
•	•	...	•	•	a	...	a	?	
a	a	...	a	•	b	...	b	?	

ARROW'S THEOREM - OPTION 2

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

■ IIA: Only order of a, b can change $\implies a \succsim b$ at the top

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
b	b	...	b	a	•	...	•	a	a
•	•	...	•	b	•	...	•	b	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	a	...	a	•	
a	a	...	a	•	b	...	b	•	

ARROW'S THEOREM - OPTION 2

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- consider c and reorder \xrightarrow{IIA} a remains at the top (it did not move)

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
•	•	...	•	a	•	...	•	a	a
•	•	...	•	c	•	...	•	?	
•	•	...	•	b	•	...	•	?	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
c	c	...	c	•	c	...	c	?	
b	b	...	b	•	a	...	a	?	
a	a	...	a	•	b	...	b	?	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

■ switch a, b for $k + 1, \dots, n \xrightarrow{\text{IIA}} a \succsim c \succsim b$

\succsim_1	\succsim_2	...	\succsim_{k-1}	\succsim_k	\succsim_{k+1}	...	\succsim_n	F	f
•	•	...	•	a	•	...	•	a	a
•	•	...	•	c	•	...	•	•	
•	•	...	•	b	•	...	•	c	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
c	c	...	c	•	c	...	c	b	
b	b	...	b	•	b	...	b	•	
a	a	...	a	•	a	...	a	•	

ARROW'S THEOREM

Arrow's theorem

A pareto efficient and IIA social welfare function $F: \mathcal{P}_{\succsim}^n \rightarrow \mathcal{P}_{\succsim}$ is dictatorial.

Proof:

- consider $(\succsim_1, \dots, \succsim_n)$ s.t. $a \succ_k b$
- change s.t. c :
 1. $a \succ_k c \succ_k b$
 2. $c \succ_i d$ for every alternative $d, i \neq k$
- order of a, c same as in previous table $\implies a \succ c$
- pareto optimality: $c \succ b$
- Transitivity of $\succ \implies a \succ b$
- the same for any combination of alternatives \implies Player k is dictator!

SOCIAL CHOICE FUNCTION

Social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is

1. (unanimous)

$$\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \dots, \prec_n) = a$$

▶ Alternative a is unanimously chosen!

2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec_i^* b) \implies$
 $(f(\prec_1, \dots, \prec_n) = a \implies f(\prec_1^*, \dots, \prec_n^*) = a)$

▶ Better preference cannot change the selection.

3. (dictatorial) $\exists k \in N : f(\prec_1, \dots, \prec_n) = a$ where $a \prec_k d$ for every $d \in A \setminus \{a\}$

▶ There is a dictator k !

Muller-Satterthwaite

A unanimous and monotonic social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is dictatorial.

Proof:

- Similar to the proof of Arrow

...AND IT GETS WORSE...

Social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is

4. (strategy-proof)

$$f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k, \succsim_{k+1}, \dots, \succsim_n) \succsim_k f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k^*, \succsim_{k+1}, \dots, \succsim_n)$$

▶ You cannot get a better result by lying

5. (surjective) $\forall a \in A, \exists (\succsim_1, \dots, \succsim_n) \in \mathcal{P}_{\succsim}^n : f(\succsim_1, \dots, \succsim_n) = a$

▶ Each alternative is possible

Gibbard-Satterthwaite

A surjective strategy-proof social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is dictatorial.

Proof: strategy-proof \implies monotonic

■ $f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k, \succsim_{k+1}, \dots, \succsim_n) = a$

■ $\forall b \in A \setminus \{a\} : a \succsim_k b \implies a \succsim_k^* b$

■ if $f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k^*, \succsim_{k+1}, \dots, \succsim_n) = c$

▶ $\xrightarrow{\text{strategy-proof}} a \succsim_k c \implies a \succsim_k^* c$ and $c \succsim_k^* a$

▶ $\xrightarrow{\text{antisymmetry}} c = a$

...AND IT GETS WORSE...

Social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is

2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \succsim_i b \implies a \succsim_i^* b) \implies$
 $(f(\succsim_1, \dots, \succsim_n) = a \implies f(\succsim_1^*, \dots, \succsim_n^*) = a)$

4. (strategy-proof)

$$f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k, \succsim_{k+1}, \dots, \succsim_n) \succsim_k f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k^*, \succsim_{k+1}, \dots, \succsim_n)$$

Gibbard-Satterthwaite

A surjective strategy-proof social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is dictatorial.

Proof: Surjective + strategy-proof \implies unanimous + monotonic

- $f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k, \succsim_{k+1}, \dots, \succsim_n) = a$
- $\forall b \in A \setminus \{a\} : a \succsim_k b \implies a \succsim_k^* b$
- then $f(\succsim_1, \dots, \succsim_{k-1}, \succsim_k^*, \succsim_{k+1}, \dots, \succsim_n) = a$
- apply for $k \in N$

...AND IT GETS WORSE...

Social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is

1. (unanimous) $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \dots, \prec_n) = a$
2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec_i^* b) \implies (f(\prec_1, \dots, \prec_n) = a \implies f(\prec_1^*, \dots, \prec_n^*) = a)$
5. (surjective) $\forall a \in A, \exists (\prec_1, \dots, \prec_n) \in \mathcal{P}_{\succsim}^n : f(\prec_1, \dots, \prec_n) = a$

Gibbard-Satterthwaite

A surjective strategy-proof social choice function $f: \mathcal{P}_{\succsim}^n \rightarrow A$ is dictatorial.

Proof: Surjective + monotonic \implies unanimous

- $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b$
- surjectivity: $\exists (\prec_1^*, \dots, \prec_n^*) : f(\prec_1^*, \dots, \prec_n^*) = a$
- monotonicity: move a to top of each players preference
 - ▶ $\implies a$ is the social choice
- monotonicity: change each players preference to \prec_i
 - ▶ $\implies a$ is the social choice

Social Choice

Social choice theory studies the aggregation of individual preferences into a common or social preference. In the classical model of social choice, there is a finite number of agents who have preferences over a finite number of alternatives.

There are two impossibility theorems, which state that under mild and natural assumptions, the only way to make a social choice is to follow a dictatorial approach - the social choice is according to single players' preferences.

There is a large literature that tries to escape the negative conclusions by adapting the model and/or restricting the domain.