Cooperative game theory

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Social Choice and Impossibility Theorems

Example of social choice

Agent	a_1	a_2	a_3	a_4	a_5
1	5	1	3	2	4
2	1	2	3	4	5
3	3	4	5	2	1

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- $\{a_1, \ldots, a_5\}$ ... alternative
- Preferences
  - **Agent 1:** $a_2 \prec_1 a_4 \prec_1 a_3 \prec_1 a_5 \prec_1 a_1$

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  - **Agent 2**: $a_1 \prec_2 a_2 \prec_2 a_3 \prec_2 a_4 \prec_2 a_5$
  - **Agent 3**: $a_5 \prec_3 a_4 \prec_3 a_1 \prec_2 a_5 \prec_3 a_3$

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  - **Agent 3**: $a_5 \prec_3 a_4 \prec_3 a_1 \prec_2 a_5 \prec_3 a_3$

Goal: *What is the best **social choice** among the alternatives?*

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Choice?					

WHAT IS THE BEST **SOCIAL CHOICE** AMONG THE ALTERNATIVES? BORDA SCORES

Agent	a_1	a_2	a_3	a_4	a_5
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$\sum$					

What is the best **social choice** among the alternatives? Borda scores

Agent	a_1	a_2	a_3	a_4	a_5
1	5	1	3	2	4
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$\sum$	9	7	11	8	10

- Social preference
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- Social preference
  - $a_2 \prec a_4 \prec a_1 \prec a_5 \prec a_3$
    - a_2 ... social choice
- Disadvantage of this approach?
  - ranking of two alternatives **may** depend **not** solely on individual preferences between the alternatives

What is the best **social choice** among the alternatives? **NOT** Borda scores

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- social preference **did**!

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- Arrow: *Only by selecting a dictator!*

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  - $(\prec_1, \ldots, \prec_n) \in \mathcal{P}_{\prec}^n$ ... strict preference profile
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  - e.g. $F(\prec_1, \ldots, \prec_n) = \preccurlyeq$

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   - *Preference of alternatives determines their social preference*
3. (Dictatorial) $\exists i \in N : F(\prec_1, \ldots, \prec_n) = \prec_i$
   - *There is a dictator!*

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$\prec_1$	$\prec_2$	$\ldots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\ldots$	$\prec_n$	F	f
a	a	$\ldots$	a	a	a	$\ldots$	a	a	a
•	•	$\ldots$	•	•	•	$\ldots$	•	•	
•	•	$\ldots$	•	•	•	$\ldots$	•	•	
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	
•	•	$\ldots$	•	•	•	$\ldots$	•	•	
•	•	$\ldots$	•	•	•	$\ldots$	•	•	
b	b	$\ldots$	b	b	b	$\ldots$	b	b	

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Proof:

- move b up in $\prec_1$

$\prec_1$	$\prec_2$	...	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	...	$\prec_n$	F	f
a	a	...	a	a	a	...	a	a	a
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
b	•	...	•	•	•	...	•	•	
•	b	...	b	b	b	...	b	b	

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a	a	$\dots$	a	a	a	$\dots$	a	a	a
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
b	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	b	
•	b	$\dots$	b	b	b	$\dots$	b	•	

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a	a	$\dots$	a	a	a	$\dots$	a	a	a
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	b	
•	b	$\dots$	b	b	b	$\dots$	b	•	

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•	•	$\dots$	•	•	•	$\dots$	•	•	
b	•	$\dots$	•	•	•	$\dots$	•	•	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	b	
•	b	$\dots$	b	b	b	$\dots$	b	•	

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a	a	…	a	a	a	…	a	a	a
b	•	…	•	•	•	…	•	•	
•	•	…	•	•	•	…	•	•	
⋮	⋮	…	⋮	⋮	⋮	…	⋮	⋮	
•	•	…	•	•	•	…	•	b	
•	•	…	•	•	•	…	•	•	
•	b	…	b	b	•	…	b	•	

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a	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
•	•	$\dots$	•	•	•	$\dots$	•	b	
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	b	$\dots$	b	b	•	$\dots$	b	•	

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$\prec_1$	$\prec_2$	...	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	...	$\prec_n$	F	f
b	a	...	a	a	a	...	a	a	a
a	●	...	●	●	●	...	●	●	
●	●	...	●	●	●	...	●	●	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
●	●	...	●	●	●	...	●	●	
●	b	...	●	●	●	...	●	●	
●	●	...	b	b	●	...	b	●	

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- move *b* up in $\prec_2$

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b	b	...	a	a	a	...	a	a	a
a	a	...	●	●	●	...	●	●	
●	●	...	●	●	●	...	●	●	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
●	●	...	●	●	●	...	●	●	
●	●	...	●	●	●	...	●	●	
●	●	...	b	b	●	...	b	●	

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Proof:

- move b up in $\prec_i$ until it holds $a \preccurlyeq b$

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	a	a	a	$\dots$	a	a	a
a	a	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	b	b	•	$\dots$	b	•	

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A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- move b up in $\prec_i$ until it holds $a \preccurlyeq b$

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	a	a	$\dots$	a	a	a
a	a	$\dots$	a	b	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	b	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	b	$\dots$	b	$\bullet$	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- move b up in $\prec_k \implies a \not\preccurlyeq b$

$\prec_1$	$\prec_2$	...	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	...	$\prec_n$	F	f
b	b	...	b	b	a	...	a	b	b
a	a	...	a	a	•	...	•	a	
•	•	...	•	•	•	...	•	•	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	b	...	b	•	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- move a to the bottom in $\prec_1$

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	b	a	$\dots$	a	b	b
•	a	$\dots$	a	a	•	$\dots$	•	a	
•	•	$\dots$	•	•	•	$\dots$	•	•	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
a	•	$\dots$	•	•	b	$\dots$	b	•	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- move a to the bottom in $\prec_2$

$\prec_1$	$\prec_2$	...	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	...	$\prec_n$	F	f
b	b	...	b	b	a	...	a	b	b
•	•	...	a	a	•	...	•	•	
•	•	...	•	•	•	...	•	a	
⋮	⋮	...	⋮	⋮	⋮	...	⋮	⋮	
•	•	...	•	•	•	...	•	•	
•	•	...	•	•	•	...	•	•	
a	a	...	•	•	b	...	b	•	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- move a to the bottom in $\prec_{k-1}$

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	b	a	$\dots$	a	b	b
•	•	$\dots$	•	a	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
•	•	$\dots$	•	•	•	$\dots$	•	•	
•	•	$\dots$	•	•	•	$\dots$	•	•	
a	a	$\dots$	a	•	b	$\dots$	b	•	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- move a just above in b $\prec_{k+1}$

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	b	$\bullet$	$\dots$	a	b	b
$\bullet$	$\bullet$	$\dots$	$\bullet$	a	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	a	$\dots$	$\bullet$	$\bullet$	
a	a	$\dots$	a	$\bullet$	b	$\dots$	b	$\bullet$	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- move a just above b in $\prec_{k+1}, \ldots, \prec_n$

$\prec_1$	$\prec_2$	$\ldots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\ldots$	$\prec_n$	F	f
b	b	$\ldots$	b	b	$\bullet$	$\ldots$	$\bullet$	b	b
$\bullet$	$\bullet$	$\ldots$	$\bullet$	a	$\bullet$	$\ldots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\ldots$	$\bullet$	$\bullet$	$\bullet$	$\ldots$	$\bullet$	$\bullet$	
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\ldots$	$\bullet$	$\bullet$	$\bullet$	$\ldots$	$\bullet$	a	
$\bullet$	$\bullet$	$\ldots$	$\bullet$	$\bullet$	a	$\ldots$	a	$\bullet$	
a	a	$\ldots$	a	$\bullet$	b	$\ldots$	b	$\bullet$	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- change $b \prec_k a$ to $a \prec_k b$

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	a	$\bullet$	$\dots$	$\bullet$	?	?
$\bullet$	$\bullet$	$\dots$	$\bullet$	b	$\bullet$	$\dots$	$\bullet$	?	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	?	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	?	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	a	$\dots$	a	?	
a	a	$\dots$	a	$\bullet$	b	$\dots$	b	?	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- IIA: Only order of a, b can change $\implies$ b (almost) at the top

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	a	$\bullet$	$\dots$	$\bullet$	b	b
$\bullet$	$\bullet$	$\dots$	$\bullet$	b	$\bullet$	$\dots$	$\bullet$	a	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	a	$\dots$	a	$\bullet$	
a	a	$\dots$	a	$\bullet$	b	$\dots$	b	$\bullet$	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- IIA: Only order of a, b can change $\implies$ b (almost) at the top

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	a	$\bullet$	$\dots$	$\bullet$	a	a
$\bullet$	$\bullet$	$\dots$	$\bullet$	b	$\bullet$	$\dots$	$\bullet$	b	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	a	$\dots$	a	$\bullet$	
a	a	$\dots$	a	$\bullet$	b	$\dots$	b	$\bullet$	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- recall previous table and move a to the bottom

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	a	a	$\dots$	a	a	a
a	a	$\dots$	a	b	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	b	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	b	$\dots$	b	$\bullet$	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- preferences between a, b do **not** change $\implies$ $a \preccurlyeq b$

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
b	b	$\dots$	b	a	$\bullet$	$\dots$	$\bullet$	?	?
$\bullet$	$\bullet$	$\dots$	$\bullet$	b	$\bullet$	$\dots$	$\bullet$	?	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	?	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	?	
$\bullet$	$\bullet$	$\dots$	$\bullet$	$\bullet$	a	$\dots$	a	?	
a	a	$\dots$	a	$\bullet$	b	$\dots$	b	?	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- IIA: Only order of a, b can change $\implies$ $a \preccurlyeq b$ at the top

$\prec_1$	$\prec_2$	…	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	…	$\prec_n$	F	f
b	b	…	b	a	•	…	•	a	a
•	•	…	•	b	•	…	•	b	
•	•	…	•	•	•	…	•	•	
⋮	⋮	…	⋮	⋮	⋮	…	⋮	⋮	
•	•	…	•	•	•	…	•	•	
•	•	…	•	•	a	…	a	•	
a	a	…	a	•	b	…	b	•	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \rightarrow \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- consider c and reorder $\overset{IIA}{\Longrightarrow}$ a remains at the top (it did not move)

$\prec_1$	$\prec_2$	$\dots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\dots$	$\prec_n$	F	f
•	•	$\dots$	•	a	•	$\dots$	•	a	a
•	•	$\dots$	•	c	•	$\dots$	•	?	
•	•	$\dots$	•	b	•	$\dots$	•	?	
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
c	c	$\dots$	c	•	c	$\dots$	c	?	
b	b	$\dots$	b	•	a	$\dots$	a	?	
a	a	$\dots$	a	•	b	$\dots$	b	?	

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- switch a, b for $k+1, \ldots, n \overset{IIA}{\Longrightarrow} a \preccurlyeq c \preccurlyeq b$

$\prec_1$	$\prec_2$	$\ldots$	$\prec_{k-1}$	$\prec_k$	$\prec_{k+1}$	$\ldots$	$\prec_n$	F	f
$\bullet$	$\bullet$	$\ldots$	$\bullet$	a	$\bullet$	$\ldots$	$\bullet$	a	a
$\bullet$	$\bullet$	$\ldots$	$\bullet$	c	$\bullet$	$\ldots$	$\bullet$	$\bullet$	
$\bullet$	$\bullet$	$\ldots$	$\bullet$	b	$\bullet$	$\ldots$	$\bullet$	c	
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	
c	c	$\ldots$	c	$\bullet$	c	$\ldots$	c	b	
b	b	$\ldots$	b	$\bullet$	b	$\ldots$	b	$\bullet$	
a	a	$\ldots$	a	$\bullet$	a	$\ldots$	a	$\bullet$	

Arrow's theorem

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Proof:

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- consider $(\prec_1, \ldots, \prec_n)$ s.t. $a \prec_k b$

Arrow's theorem

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Proof:

- consider $(\prec_1, \ldots, \prec_n)$ s.t. $a \prec_k b$
- change s.t. c:

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- consider $(\prec_1, \ldots, \prec_n)$ s.t. $a \prec_k b$
- change s.t. c:
  1. $a \prec_k c \prec_k b$

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- consider $(\prec_1, \ldots, \prec_n)$ s.t. $a \prec_k b$
- change s.t. c:
  1. $a \prec_k c \prec_k b$
  2. $c \prec_i d$ for every alternative d, $i \neq k$

Arrow's theorem

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Proof:

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- change s.t. c:
  1. $a \prec_k c \prec_k b$
  2. $c \prec_i d$ for every alternative d, $i \neq k$
- order of a, c same as in previous table $\implies a \preccurlyeq c$

Arrow's theorem

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Proof:

- consider $(\prec_1, \ldots, \prec_n)$ s.t. $a \prec_k b$
- change s.t. c:
  1. $a \prec_k c \prec_k b$
  2. $c \prec_i d$ for every alternative d, $i \neq k$
- order of a, c same as in previous table $\implies a \preccurlyeq c$
- pareto optimality: $c \preccurlyeq b$

Arrow's theorem

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Proof:

- consider $(\prec_1, \ldots, \prec_n)$ s.t. $a \prec_k b$
- change s.t. c:
  1. $a \prec_k c \prec_k b$
  2. $c \prec_i d$ for every alternative d, $i \neq k$
- order of a, c same as in previous table $\implies a \preccurlyeq c$
- pareto optimality: $c \preccurlyeq b$
- Transitivity of $\preccurlyeq \implies a \preccurlyeq b$

Arrow's theorem

A pareto efficient and IIA social welfare function $F\colon \mathcal{P}_{\prec}^{n} \to \mathcal{P}_{\preccurlyeq}$ is dictatorial.

Proof:

- consider $(\prec_1, \ldots, \prec_n)$ s.t. $a \prec_k b$
- change s.t. c:
  1. $a \prec_k c \prec_k b$
  2. $c \prec_i d$ for every alternative d, $i \neq k$
- order of a, c same as in previous table $\implies a \preccurlyeq c$
- pareto optimality: $c \preccurlyeq b$
- Transitivity of $\preccurlyeq \implies a \preccurlyeq b$
- the same for any combination of alternatives $\implies$ Player k is dictator!

Social choice function

Social choice function $f\colon \mathcal{P}_{\prec}^{n} \to A$ is

Social choice function $f\colon \mathcal{P}_{\prec}^{n} \to A$ is

1. (unanimous)
   $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \ldots, \prec_n) = a$
   - *Alternative a is unanimously chosen!*

Social choice function

Social choice function $f\colon \mathcal{P}^n_{\prec} \to A$ is

1. (unanimous)
   $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \ldots, \prec_n) = a$
   - *Alternative a is unanimously chosen!*
2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec^*_i b) \implies (f(\prec_1, \ldots, \prec_n) = a \implies f(\prec^*_1, \ldots, \prec^*_n) = a)$
   - *Better preference cannot change the selection.*

Social choice function $f : \mathcal{P}^n_{\prec} \to A$ is

1. (unanimous)
   $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \ldots, \prec_n) = a$
   - *Alternative a is unanimously chosen!*
2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec^*_i b) \implies (f(\prec_1, \ldots, \prec_n) = a \implies f(\prec^*_1, \ldots, \prec^*_n) = a)$
   - *Better preference cannot change the selection.*
3. (dictatorial) $\exists k \in N : f(\prec_1, \ldots, \prec_n) = a$ where $a \prec_k d$ for every $d \in A \setminus \{a\}$
   - *There is a dictator k!*

Here we go again...

Muller-Satterthwaite

A unanimous and monotonic social choice function $f : \mathcal{P}_{\prec}^{n} \to A$ is dictatorial.

Proof:

Muller-Satterthwaite

A unanimous and monotonic social choice function $f : \mathcal{P}_{\prec}^{n} \rightarrow A$ is dictatorial.

Proof:

- Similar to the proof of Arrow

…AND IT GETS WORSE…

Social choice function $f\colon \mathcal{P}_{\prec}^{n} \to A$ is

Social choice function $f\colon \mathcal{P}^n_{\prec} \to A$ is

4. (strategy-proof)

   $f(\prec_1,\ldots,\prec_{k-1},\prec_k,\prec_{k+1},\ldots,\prec_n) \prec_k f(\prec_1,\ldots,\prec_{k-1},\prec_k^*,\prec_{k+1},\ldots,\prec_n)$

   - *You cannot get a better result by lying*

...AND IT GETS WORSE...

Social choice function $f: \mathcal{P}_{\prec}^{n} \to A$ is

4. (strategy-proof)

   $f(\prec_1, \ldots, \prec_{k-1}, \prec_k, \prec_{k+1}, \ldots, \prec_n) \prec_k f(\prec_1, \ldots, \prec_{k-1}, \prec_k^*, \prec_{k+1}, \ldots, \prec_n)$
   - *You cannot get a better result by lying*
5. (surjective) $\forall a \in A, \exists(\prec_1, \ldots, \prec_n) \in \mathcal{P}_{\prec}^{n} : f(\prec_1, \ldots, \prec_n) = a$
   - *Each alternative is possible*

Social choice function $f : \mathcal{P}^n_{\prec} \to A$ is

4. (strategy-proof)

   $f(\prec_1, \ldots, \prec_{k-1}, \prec_k, \prec_{k+1}, \ldots, \prec_n) \prec_k f(\prec_1, \ldots, \prec_{k-1}, \prec^*_k, \prec_{k+1}, \ldots, \prec_n)$
   - *You cannot get a better result by lying*
5. (surjective) $\forall a \in A, \exists(\prec_1, \ldots, \prec_n) \in \mathcal{P}^n_{\prec} : f(\prec_1, \ldots, \prec_n) = a$
   - *Each alternative is possible*

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_{\prec} \to A$ is dictatorial.

Social choice function $f : \mathcal{P}^n_{\prec} \to A$ is

4. (strategy-proof)
   $f(\prec_1,\ldots,\prec_{k-1},\prec_k,\prec_{k+1},\ldots,\prec_n) \prec_k f(\prec_1,\ldots,\prec_{k-1},\prec_k^*,\prec_{k+1},\ldots,\prec_n)$
   - *You cannot get a better result by lying*
5. (surjective) $\forall a \in A, \exists(\prec_1, \ldots, \prec_n) \in \mathcal{P}^n_{\prec} : f(\prec_1, \ldots, \prec_n) = a$
   - *Each alternative is possible*

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_{\prec} \to A$ is dictatorial.

Proof: strategy-proof $\implies$ monotonic

Social choice function $f : \mathcal{P}^n_\prec \to A$ is

4. (strategy-proof)

   $f(\prec_1, \dots, \prec_{k-1}, \prec_k, \prec_{k+1}, \dots, \prec_n) \prec_k f(\prec_1, \dots, \prec_{k-1}, \prec_k^*, \prec_{k+1}, \dots, \prec_n)$
   - *You cannot get a better result by lying*
5. (surjective) $\forall a \in A, \exists(\prec_1, \dots, \prec_n) \in \mathcal{P}^n_\prec : f(\prec_1, \dots, \prec_n) = a$
   - *Each alternative is possible*

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_\prec \to A$ is dictatorial.

Proof: strategy-proof $\implies$ monotonic

- $f(\prec_1, \dots, \prec_{k-1}, \prec_k, \prec_{k+1}, \dots, \prec_n) = a$
- $\forall b \in A \setminus \{a\} : a \prec_k b \implies a \prec_k^* b$

Social choice function $f : \mathcal{P}_{\prec}^{n} \to A$ is

4. (strategy-proof)

   $f(\prec_1,\dots,\prec_{k-1},\prec_k,\prec_{k+1},\dots,\prec_n) \prec_k f(\prec_1,\dots,\prec_{k-1},\prec_k^*,\prec_{k+1},\dots,\prec_n)$
   - *You cannot get a better result by lying*
5. (surjective) $\forall a \in A, \exists(\prec_1, \dots, \prec_n) \in \mathcal{P}_{\prec}^{n} : f(\prec_1, \dots, \prec_n) = a$
   - *Each alternative is possible*

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}_{\prec}^{n} \to A$ is dictatorial.

Proof: strategy-proof $\implies$ monotonic

- $f(\prec_1, \dots, \prec_{k-1}, \prec_k, \prec_{k+1}, \dots, \prec_n) = a$
- $\forall b \in A \setminus \{a\} : a \prec_k b \implies a \prec_k^* b$
- if $f(\prec_1, \dots, \prec_{k-1}, \prec_k^*, \prec_{k+1}, \dots, \prec_n) = c$

Social choice function $f\colon \mathcal{P}^n_{\prec} \to A$ is

4. (strategy-proof)
   $f(\prec_1,\ldots,\prec_{k-1},\prec_k,\prec_{k+1},\ldots,\prec_n) \prec_k f(\prec_1,\ldots,\prec_{k-1},\prec^*_k,\prec_{k+1},\ldots,\prec_n)$
   - *You cannot get a better result by lying*
5. (surjective) $\forall a \in A, \exists(\prec_1,\ldots,\prec_n) \in \mathcal{P}^n_{\prec} : f(\prec_1,\ldots,\prec_n) = a$
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Proof: strategy-proof $\implies$ monotonic

- $f(\prec_1,\ldots,\prec_{k-1},\prec_k,\prec_{k+1},\ldots,\prec_n) = a$
- $\forall b \in A \setminus \{a\} : a \prec_k b \implies a \prec^*_k b$
- if $f(\prec_1,\ldots,\prec_{k-1},\prec^*_k,\prec_{k+1},\ldots,\prec_n) = c$
  - $\overset{strategy-proof}{\Longrightarrow} a \prec_k c \implies a \prec^*_k c$ and $c \prec^*_k a$

Social choice function $f : \mathcal{P}^n_{\prec} \to A$ is

4. (strategy-proof)

   $f(\prec_1,\ldots,\prec_{k-1},\prec_k,\prec_{k+1},\ldots,\prec_n) \prec_k f(\prec_1,\ldots,\prec_{k-1},\prec_k^*,\prec_{k+1},\ldots,\prec_n)$
   - *You cannot get a better result by lying*
5. (surjective) $\forall a \in A, \exists(\prec_1,\ldots,\prec_n) \in \mathcal{P}^n_{\prec} : f(\prec_1,\ldots,\prec_n) = a$
   - *Each alternative is possible*

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Proof: strategy-proof $\implies$ monotonic

- $f(\prec_1,\ldots,\prec_{k-1},\prec_k,\prec_{k+1},\ldots,\prec_n) = a$
- $\forall b \in A \setminus \{a\} : a \prec_k b \implies a \prec_k^* b$
- if $f(\prec_1,\ldots,\prec_{k-1},\prec_k^*,\prec_{k+1},\ldots,\prec_n) = c$
  - $\overset{strategy-proof}{\implies} a \prec_k c \implies a \prec_k^* c$ and $c \prec_k^* a$
  - $\overset{antisymmetry}{\implies} c = a$

...AND IT GETS WORSE...

Social choice function $f : \mathcal{P}_{\prec}^n \to A$ is

2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec_i^* b) \implies (f(\prec_1, \ldots, \prec_n) = a \implies f(\prec_1^*, \ldots, \prec_n^*) = a)$
4. (strategy-proof) $f(\prec_1,\ldots,\prec_{k-1},\prec_k,\prec_{k+1},\ldots,\prec_n) \prec_k f(\prec_1,\ldots,\prec_{k-1},\prec_k^*,\prec_{k+1},\ldots,\prec_n)$

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}_{\prec}^n \to A$ is dictatorial.

Proof: Surjective + strategy-proof $\implies$ unanimous + monotonic

- $f(\prec_1, \ldots, \prec_{k-1}, \prec_k, \prec_{k+1}, \ldots, \prec_n) = a$
- $\forall b \in A \setminus \{a\} : a \prec_k b \implies a \prec_k^* b$
- then $f(\prec_1, \ldots, \prec_{k-1}, \prec_k^*, \prec_{k+1}, \ldots, \prec_n) = a$

Social choice function $f : \mathcal{P}^n_\prec \to A$ is

2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec^*_i b) \implies (f(\prec_1, \ldots, \prec_n) = a \implies f(\prec^*_1, \ldots, \prec^*_n) = a)$
4. (strategy-proof) $f(\prec_1, \ldots, \prec_{k-1}, \prec_k, \prec_{k+1}, \ldots, \prec_n) \prec_k f(\prec_1, \ldots, \prec_{k-1}, \prec^*_k, \prec_{k+1}, \ldots, \prec_n)$

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_\prec \to A$ is dictatorial.

Proof: Surjective + strategy-proof $\implies$ unanimous + monotonic

- $f(\prec_1, \ldots, \prec_{k-1}, \prec_k, \prec_{k+1}, \ldots, \prec_n) = a$
- $\forall b \in A \setminus \{a\} : a \prec_k b \implies a \prec^*_k b$
- then $f(\prec_1, \ldots, \prec_{k-1}, \prec^*_k, \prec_{k+1}, \ldots, \prec_n) = a$
- apply for $k \in N$

Social choice function $f : \mathcal{P}_{\prec}^{n} \to A$ is

1. (unanimous) $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \ldots, \prec_n) = a$
2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec_i^* b) \implies (f(\prec_1, \ldots, \prec_n) = a \implies f(\prec_1^*, \ldots, \prec_n^*) = a)$
5. (surjective) $\forall a \in A, \exists (\prec_1, \ldots, \prec_n) \in \mathcal{P}_{\prec}^{n} : f(\prec_1, \ldots, \prec_n) = a$

Social choice function $f : \mathcal{P}^n_{\prec} \to A$ is

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Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_{\prec} \to A$ is dictatorial.

...AND IT GETS WORSE...

Social choice function $f : \mathcal{P}^n_\prec \to A$ is

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Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_\prec \to A$ is dictatorial.

Proof: Surjective + monotonic $\implies$ unanimous

- $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b$

...AND IT GETS WORSE...

Social choice function $f : \mathcal{P}^n_{\prec} \to A$ is

1. (unanimous) $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \ldots, \prec_n) = a$
2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec_i^* b) \implies (f(\prec_1, \ldots, \prec_n) = a \implies f(\prec_1^*, \ldots, \prec_n^*) = a)$
5. (surjective) $\forall a \in A, \exists(\prec_1, \ldots, \prec_n) \in \mathcal{P}^n_{\prec} : f(\prec_1, \ldots, \prec_n) = a$

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_{\prec} \to A$ is dictatorial.

Proof: Surjective + monotonic $\implies$ unanimous

- $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b$
- surjectivity: $\exists(\prec_1^*, \ldots, \prec_n^*) : f(\prec_1^*, \ldots, \prec_n^*) = a$

Social choice function $f: \mathcal{P}^n_{\prec} \to A$ is

1. (unanimous) $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \ldots, \prec_n) = a$
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Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f: \mathcal{P}^n_{\prec} \to A$ is dictatorial.

Proof: Surjective + monotonic $\implies$ unanimous

- $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b$
- surjectivity: $\exists(\prec_1^*, \ldots, \prec_n^*) : f(\prec_1^*, \ldots, \prec_n^*) = a$
- monotonicity: move a to top of each players preference
  - $\implies a$ is the social choice

...AND IT GETS WORSE...

Social choice function $f : \mathcal{P}^n_{\prec} \to A$ is

1. (unanimous) $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b \implies f(\prec_1, \ldots, \prec_n) = a$
2. (monotonic) $\forall i \in N, b \in A \setminus \{a\} : (a \prec_i b \implies a \prec^*_i b) \implies (f(\prec_1, \ldots, \prec_n) = a \implies f(\prec^*_1, \ldots, \prec^*_n) = a)$
5. (surjective) $\forall a \in A, \exists(\prec_1, \ldots, \prec_n) \in \mathcal{P}^n_{\prec} : f(\prec_1, \ldots, \prec_n) = a$

Gibbart-Satterthwaite

A surjective strategy-proof social choice function $f : \mathcal{P}^n_{\prec} \to A$ is dictatorial.

Proof: Surjective + monotonic $\implies$ unanimous

- $\forall i \in N, b \in A \setminus \{a\} : a \prec_i b$
- surjectivity: $\exists(\prec^*_1, \ldots, \prec^*_n) : f(\prec^*_1, \ldots, \prec^*_n) = a$
- monotonicity: move a to top of each players preference
  - $\implies$ a is the social choice
- monotonicity: change each players preference to $\prec_i$
  - $\implies$ a is the social choice

Summary

Social Choice

Social choice theory studies the aggregation of individual preferences into a common or social preference. In the classical model of social choice, there is a finite number of agents who have preferences over a finite number of alternatives.
There are two impossibility theorems, which state that under mild and natural assumptions, the only way to make a social choice is to follow a dictatorial approach - the social choice is according to single players' preferences.
There is a large literature that tries to escape the negative conclusions by adapting the model and/or restricting the domain.