Cooperative game theory

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Over-Provisining and Atomic Selfish Routing

- cheap to add additional capacity (*over-provision*)
- $\implies$ we often do so
  - to anticipate future growth in demand
  - performance reasons
    - fewer packet drops
    - fewer delays
- empirically observed

Goal: *Show this has support in the theory*

Selfish routing network

- $G = (V, E)$ ... directed graph
  - o ... origin
  - d ... destination
- r units of traffic
- $c_e \colon \mathbb{R}_+ \to \mathbb{R}_+$ ... cost function of edge e
  - monotone and non-decreasing

Communication network

- $c_e(x) = \begin{cases} \frac{1}{u_e - x} & x \leq u_e, \\ +\infty & x > u_e. \end{cases}$
  - u_e ... capacity of e
  - $c_e(x)$ ... expected per-unit delay in $M/M/1$ queue
    - from *Queueling theory* $\sim$ study of waiting lines and queues

Communication networks model

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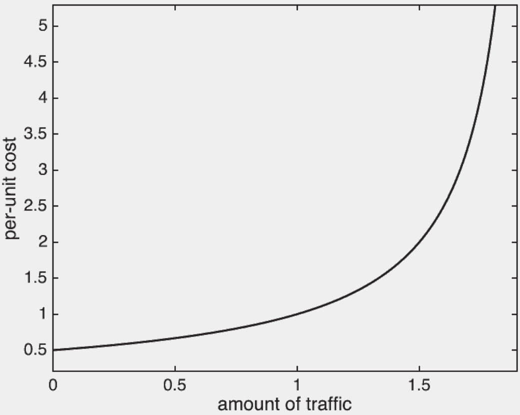


β-over-provisioned network

A selfish routing network is β-*over-provisioned* if

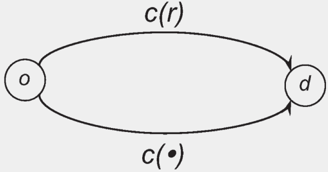
$$f_e \leq (1 - \beta)u_e,$$

where $\beta \in (0, 1)$ and f is some equilibrium flow.

- $f_e \leq u_e$ ... restriction to capacity
- $f_e \leq u_e - \beta u_e$
  - β ... represents, what portion of u_e *exceeds demand*
  - $\beta \to 0$
    - $\implies f_e \leq u_e$ ... almost all capacity used
  - $\beta \to 1$
    - $\implies f_e \leq \varepsilon$ ... almost no capcity used
  - $(1 - \beta) \cdot 100\%$ ... maximum edge utilisation under f

Pigou-like network

- A traffic rate $r \geq 0$
- A cost function $c(\bullet)$ on the lower edge
- A cost function $c(r)$ on the upper edge



- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$

Tight POA bounds for selfish routing

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

Tight POA bounds for selfish routing

For β-over-provisioned communication networks, the POA is at most $\frac{1}{2}\left(1+\sqrt{\frac{1}{\beta}}\right)$.

- $\beta \to 0$
  - not over-provisioned
  - POA $\to +\infty$
- $\beta \to 1$
  - well over-provisioned
  - POA $\to 1$

Traffic augmentation bounds

Idea: *If you raise the traffic rate, the travel time also rises.*

- compare:
  1. equilibrium flow with traffic rate r
  2. optimum-flow with traffic rate $\varepsilon \cdot r$
     - $\varepsilon \geq 1$
- alternatively:
  1. equilibrium flow with costs $\frac{c_e(\frac{x}{\varepsilon})}{\varepsilon}$
     - *faster* network
  2. optimum-flow with costs $c_e(x)$
     - the same network
- $c_e(x) = \frac{1}{u_e - x}$
  - ▶ corresponds to doubling the capacity

Resource augmentation bounds

For every selfish routing network and $r > 0$, the cost of an equilibrium flow with traffic rate r is at most the cost of the optimal flow with traffic rate $2r$.

Proof:

- f ... equilibirum flow with traffic r
- f^* ... optimal flow with traffic $2r$
- $c(f) = \sum_{e \in E} f_e c_e(f_e) = \sum_{P \in \mathcal{P}} f_P c_P(f)$
  - $\sum_{P \in \mathcal{P}} f_P = r$
  - $c_P(f) = L$ for every $f_P > 0$
- $c(f^*) =?$
  - $\implies$ analyse f^* on network with $c_e(\bullet) = c_e(f_e)$

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- $c(f) = \sum_{e \in E} f_e c_e(f_e) = \sum_{P \in \mathcal{P}} f_P c_P(f) = r \cdot L$
- $\sum_{e \in E} f_e^* c_e(f_e) = \sum_{P \in \mathcal{P}} f_P^* c_P(f)$
  - $\sum_{P \in \mathcal{P}} f_P^* = 2r$
  - $c_P(f) \geq L$ for every $P \in \mathcal{P}$

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- $c(f) = r \cdot L$
- $\sum_{e \in E} f_e c_e(f_e) = r \cdot L$
- $\sum_{e \in E} f_e^* c_e(f_e) \geq 2 \cdot L$
- $\sum_{e \in E} f_e^* c_e(f_e) - \sum_{e \in E} f_e c_e(f_e) \geq c(f)$

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- $c(f) = \sum_{e \in E} f_e^* c_e(f_e) \overset{?}{\geq} \sum_{e \in E} f_e^* c_e(f_e^*) - \sum_{e \in E} f_e c_e(f_e) \geq c(f)$
- we show: $f_e^* c_e(f_e^*) \geq f_e^* c_e(f_e) - f_e c_e(f_e)$
- or: $f_e c_e(f_e) \geq f_e^* c_e(f_e) - f_e^* c_e(f_e^*)$
    1. $f_e^* \geq f_e$
        - $c_e(f_e^*) \geq c_e(f_e)$ ... $c_e(\bullet)$ is non-decreasing
        - $0 \geq c_e(f_e) - c_e(f_e^*)$
        - $0 \geq f_e^* c_e(f_e) - f_e^* c_e(f_e^*)$
        - $f_e c_e(f_e) \geq 0$ ... $c_e(\bullet)$ is non-negative

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  2. $f_e^* < f_e$
     - $f_e c_e(f_e) \geq f_e^* c_e(f_e)$ ... $c_e(\bullet)$ is non-decreasing
     - $f_e^* c_e(f_e^*) \geq 0$ ... $c_e(\bullet)$ is non-negative
- $c(f) \geq c(f)$

Atomic selfish routing

1. **non-atomic routing** $\sim$ agents of *negligible* size
   - cars on highway, packets on communication network, ...
2. **atomic routing** $\sim$ agents control *significant portion* of traffic
   - dedicating portions of Internet traffic to end users

Atomic selfish routing network

- $1, \ldots, k$ ... agents
- $G = (V, E)$ ... directed graph
  - o_i ... origin of i
  - d_i ... destination of i
- each player controls traffic 1 on single $o_i - d_i$ path
  - traffic r_i could be defined, multiple paths could be defined
- $c_e : \mathbb{R}_+ \to \mathbb{R}_+$ ... cost function of edge e
  - monotone and non-decreasing
- $\mathcal{P}_i$ ... $o_i - d_i$ paths in G
- $(P_1, \ldots, P_k)$ ... *flow*
  - $P_i \in \mathcal{P}_i$
- Cost of flow
  - $c_{P_i}(f) = \sum_{e \in P_i} c_e(f_e)$
  - $C(f) = \sum_{e \in E} f_e \cdot c_e(f_e) = \sum_{i=1}^{k} \sum_{P_i \in \mathcal{P}_i} f_{P_i} \cdot c_{P_i}(f)$

Equilibrium flow

No player can decrease their cost via a unilateral deviation

Equilibrium flow

A flow $(P_1, \ldots, P_k)$ is an *equilibrium* if, for every agent i and path $\hat{P}_i \in \mathcal{P}_i$,

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in \hat{P}_i \cap P_i} c_e(f_e) + \sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1).$$

- $\sum_{e \in P_i} c_e(f_e)$ … before deviation
- $\sum_{e \in \hat{P}_i \cap P_i} c_e(f_e) + \sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1)$ … after deviation
  - differs from definition for nonatomic case
  - we increase the cost of newly used edge

Example: Pigou-like network with 2 agents

Equilibrium flow

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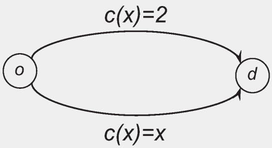


1. Optimal flow:
   - 1 agent on each edge
2. Equilibrium flow:
   2.1 1 agent on each edge
   2.2 both agents on the lower edge

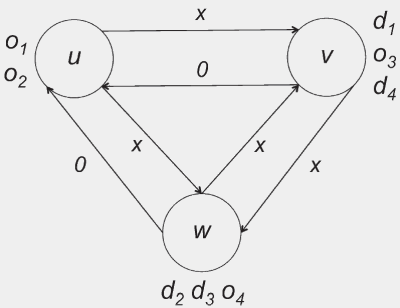
Different equilibria - different costs

- **non-atomic routing**
  - different equilibria $\Rightarrow$ same costs
- **atomic routing**
  - different equilibria $\not\Rightarrow$ same costs

Price of Anarchy (POA)

The price of anarchy of the atomic selfish routing network is the ratio between the worst-case equilibrium and the minumum possible average travel time.

Example: Bidirected triangle network



- POA?
- optimal total time:
  - one-hop paths
  - four edges with x
- equilibrium time:
  1. one-hop paths
  2. two-hop paths

POA Bound for Atomic Selfish Routing

In every atomic selfish routing network with affine cost functions, the POA is $\frac{5}{2}$.

Proof:

- f^* ... minimal cost flow
- f ... equilibrium flow
- $c_e(x) = a_e x + b_e$
  - $a_e, b_e \geq 0$
- Goal: $C(f) \leq \frac{5}{2} C(f^*)$
- **key: we use the definition of** f
  - where P_i^* is the path in f^*
- $\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in P_i^* \cap P_i} c_e(f_e) + \sum_{e \in P_i^* \setminus P_i} c_e(f_e + 1)$

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- $\leq \sum_{i=1}^{k} \sum_{e \in P_i^*} c_e(f_e + 1)$
  - $c_e(\bullet)$ ... non-decreasing
- $= \sum_{e \in E} f_e^* c_e(f_e + 1)$
- $= \sum_{e \in E} [a_e f_e^* (f_e + 1) + b_e f_e^*]$

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- $C(f) \leq \sum_{e \in E} [a_e f_e^* (f_e + 1) + b_e f_e^*]$

Disentangling lemma

For every $y, z \in \mathbb{N}$, it holds $y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$.

POA Bound for Atomic Selfish Routing

In every atomic selfish routing network with affine cost functions, the POA is $\frac{5}{2}$.

Proof: Goal: $C(f) \leq \frac{5}{2}C(f^*)$

- f^* ... minimal cost flow
- f ... equilibrium flow
- $c_e(x) = a_e x + b_e$
  - $a_e, b_e \geq 0$
- $C(f) \leq \sum_{e\in E} \left[a_e \left(\frac{5}{3}(f_e^*)^2 + \frac{1}{3}{f_e}^2\right) + b_e {f_e}^*\right]$
- $C(f) \leq \frac{5}{3}\left[\sum_{e\in E} f_e^* \left(a_e f_e^* + b_e\right)\right] + \frac{1}{3}\sum_{e\in E} a_e {f_e}^2$
- $C(f) \leq \frac{5}{3}C(f^*) + \frac{1}{3}C(f)$
- $\frac{2}{3}C(f) \leq \frac{5}{3}C(f^*)$
- $C(f) \leq \frac{3}{2}\cdot\frac{5}{3}C(f^*) = \frac{5}{2}C(f^*)$

Recall...

POA Bound for Atomic Selfish Routing

In every **non-atomic** selfish routing network with affine cost functions, the POA is $\frac{4}{3}$.

...and compare with...

POA Bound for Atomic Selfish Routing

In every **atomic** selfish routing network with affine cost functions, the POA is $\frac{5}{2}$.

- $\implies$ higher POA in atomic selfish routing networks

Over-Provisioning and Atomic Selfish Routing

In selfish routing network, the over-provisioning leads to decrease of POA. We can further modify the model of selfish routing network to **atomic** selfish routing networks, where the main difference is that each agent owns a *non-negligible* portion of traffic. For these networks, the equilibrium flow do not have to have the same costs and the POA in these networks might be higher than in non-atomic selfish routing networks.