Cooperative game theory

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Over-Provisining and Atomic Selfish Routing

Communication networks

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Goal: *Show this has support in the theory*

Selfish routing network

- $G = (V, E)$ ... directed graph
  - o ... origin
  - d ... destination
- r units of traffic
- $c_e : \mathbb{R}_+ \to \mathbb{R}_+$ ... cost function of edge e
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  - u_e ... capacity of e
  - $c_e(x)$ ... expected per-unit delay in $M/M/1$ queue
    - from *Queueling theory* $\sim$ study of waiting lines and queues

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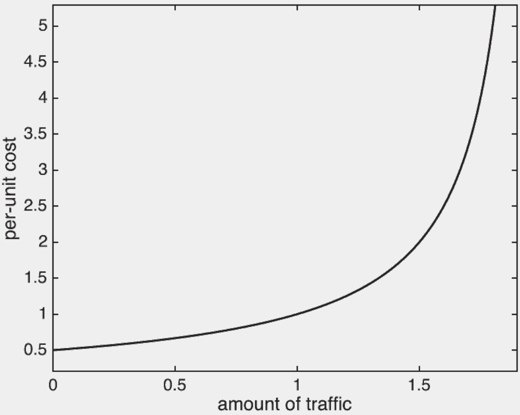

β-OVER-PROVISIONED NETWORK

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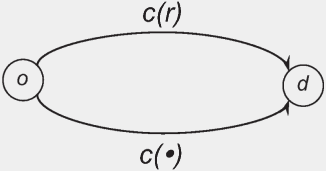
  - $(1-\beta) \cdot 100\%$ ... maximum edge utilisation under f

Pigou-like network

- A traffic rate $r \geq 0$
- A cost function $c(\bullet)$ on the lower edge
- A cost function $c(r)$ on the upper edge



- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$

Tight POA bounds for selfish routing

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

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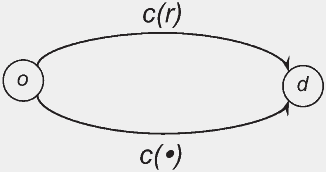

- Pigou bound: $\alpha(\mathcal{C}) = \frac{1}{2}\left(1 + \sqrt{\frac{1}{\beta}}\right)$

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  - corresponds to doubling the capacity

Resource augmentation theorem

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For every selfish routing network and $r > 0$, the cost of an equilibrium flow with traffic rate r is at most the cost of the optimal flow with traffic rate $2r$.

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  - $\implies$ analyse f^* on network with $c_e(\bullet) = c_e(f_e)$

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        - $c_e(f_e^*) \geq c_e(f_e)$ ... $c_e(\bullet)$ is non-decreasing
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Resource augmentation bounds

For every selfish routing network and $r > 0$, the cost of an equilibrium flow with traffic rate r is at most the cost of the optimal flow with traffic rate $2r$.

Proof:

- f ... equilibirum flow with traffic r
- f^* ... optimal flow with traffic $2r$
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- $c(f) \geq c(f)$

1. **non-atomic routing** $\sim$ agents of *negligible* size
   - cars on highway, packets on communication network, ...
2. **atomic routing** $\sim$ agents control *significant portion* of traffic
   - dedicating portions of Internet traffic to end users

Atomic selfish routing network

- $1, \ldots, k$ ... agents

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- $(P_1, \ldots, P_k)$ ... *flow*
  - $P_i \in \mathcal{P}_i$
- Cost of flow
  - $c_{P_i}(f) = \sum_{e \in P_i} c_e(f_e)$
  - $C(f) = \sum_{e \in E} f_e \cdot c_e(f_e) = \sum_{i=1}^{k} \sum_{P_i \in \mathcal{P}_i} f_{P_i} \cdot c_{P_i}(f)$

Equilibrium flow

No player can decrease their cost via a unilateral deviation

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Equilibrium flow

A flow $(P_1, \ldots, P_k)$ is an *equilibrium* if, for every agent i and path $\hat{P}_i \in \mathcal{P}_i$,

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- $\sum_{e \in \hat{P}_i \cap P_i} c_e(f_e) + \sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1)$ ... after deviation
  - differs from definition for nonatomic case
  - we increase the cost of newly used edge

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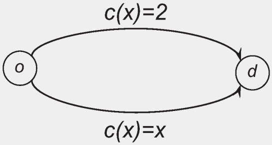

1. Optimal flow:

2. Equilibrium flow:

Example: Pigou-like network with 2 agents

Equilibrium flow

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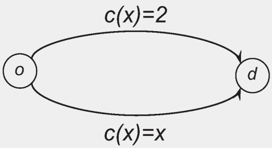

1. Optimal flow: 3
   - 1 agent on each edge
2. Equilibrium flow:

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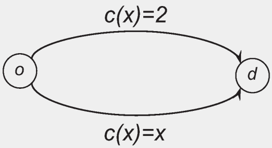

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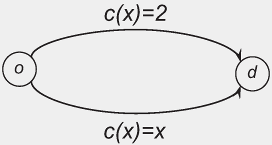

1. Optimal flow: 3
   - 1 agent on each edge
2. Equilibrium flow: 3, 4
   2.1 1 agent on each edge
   2.2 both agents on the lower edge

Different equilibria - different costs

- **non-atomic routing**
- **atomic routing**

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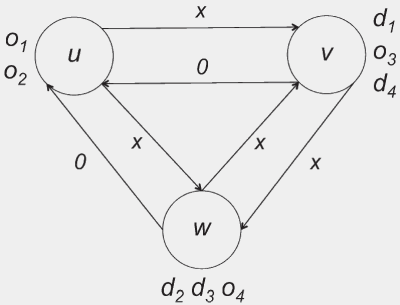
Price of Anarchy (POA)

The price of anarchy of the atomic selfish routing network is the ratio between the worst-case equilibrium and the minumum possible average travel time.

Example: Bidirected triangle network



- POA?

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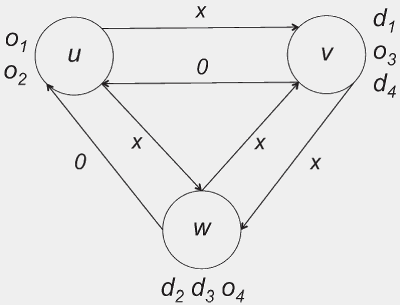

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- optimal total time:
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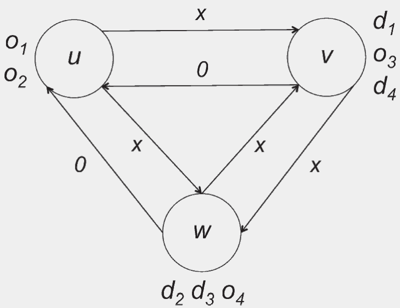

- POA?
- optimal total time: 4
  - one-hop paths
  - four edges with x
- equilibrium time:

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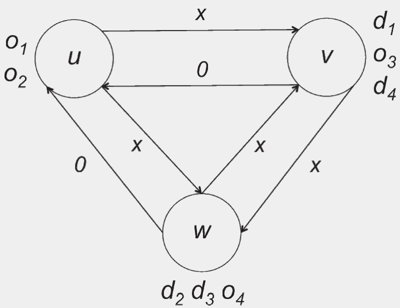

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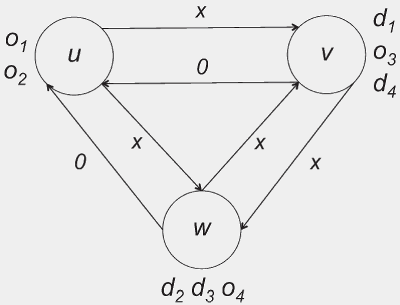

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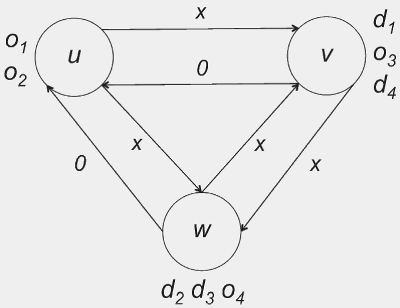

- POA? $\frac{10}{4} = 2.5$
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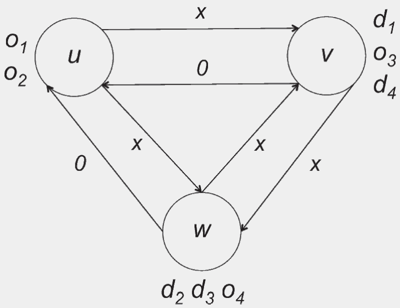

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POA Bound for Atomic Selfish Routing

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In every atomic selfish routing network with affine cost functions, the POA is $\frac{5}{2}$.

Proof:

- f^* ... minimal cost flow
- f ... equilibrium flow
- $c_e(x) = a_e x + b_e$
  - $a_e, b_e \geq 0$
- Goal: $C(f) \leq \frac{5}{2} C(f^*)$
- **key: we use the definition of** f
  - where P_i^* is the path in f^*
- $\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in P_i^* \cap P_i} c_e(f_e) + \sum_{e \in P_i^* \setminus P_i} c_e(f_e + 1)$

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- $= \sum_{e \in E} [a_e f_e^* (f_e + 1) + b_e f_e^*]$

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Disentangling lemma

For every $y, z \in \mathbb{N}$, it holds $y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$.

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- $C(f) \leq \frac{5}{3} \left[\sum_{e \in E} f_e^* \left(a_e f_e^* + b_e \right) \right] + \frac{1}{3} \sum_{e \in E} a_e {f_e}^2$

POA Bound for Atomic Selfish Routing

In every atomic selfish routing network with affine cost functions, the POA is $\frac{5}{2}$.

Proof: Goal: $C(f) \leq \frac{5}{2}C(f^*)$

- f^* ... minimal cost flow
- f ... equilibrium flow
- $c_e(x) = a_e x + b_e$
  - $a_e, b_e \geq 0$
- $C(f) \leq \sum_{e \in E} \left[a_e \left(\frac{5}{3}(f_e^*)^2 + \frac{1}{3}{f_e}^2 \right) + b_e {f_e}^* \right]$
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- $C(f) \leq \frac{3}{2} \cdot \frac{5}{3} C(f^*) = \frac{5}{2} C(f^*)$

POA Bound for Atomic Selfish Routing

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- $\Longrightarrow$ higher POA in atomic selfish routing networks

Over-Provisioning and Atomic Selfish Routing

In selfish routing network, the over-provisioning leads to decrease of POA. We can further modify the model of selfish routing network to **atomic** selfish routing networks, where the main difference is that each agent owns a *non-negligible* portion of traffic. For these networks, the equilibrium flow do not have to have the same costs and the POA in these networks might be higher than in non-atomic selfish routing networks.