Cooperative game theory

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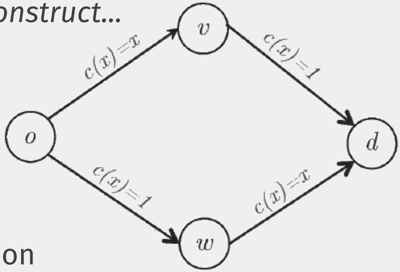
Selfish routing and POA

Goal: *Construction of traffic network resistant to selfish drivers*

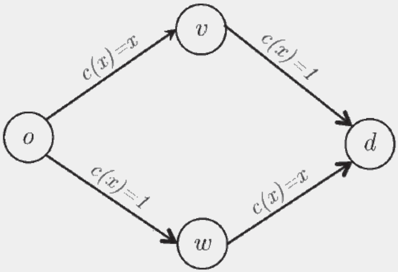
Question: *If we construct...*



- o ... origin
- d ... destination
- $o \rightarrow v \rightarrow d$, $o \rightarrow w \rightarrow d$ ... routes
- wide road
  - $c(x) = 1$ ... travel time is always 1 hour
- narrow road
  - $c(x) = x$ ... travel time is proportional to the amount traffic

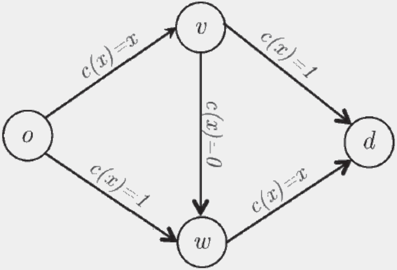
...what amount of traffic to expect on each route?

Question: *What amount of traffic to expect on each route?*



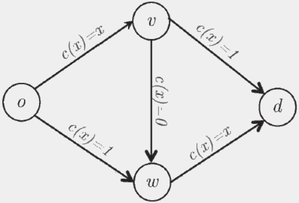
- both routes are identical $\implies$ traffic splits equally
- **Answer**: Each driver's travel time is $1 + \frac{1}{2} = \frac{3}{2}$ hours

Question: *What amount of traffic to expect on each route?*



- new route $o \to v \to w \to d$
- it is always better than $o \to v \to d$ or $o \to w \to d$
- $\implies$ all drivers prefer this route
- $\implies$ **each driver drives** $1 + 1 = 2$ **hours!**

Question: *How much worse is this?*

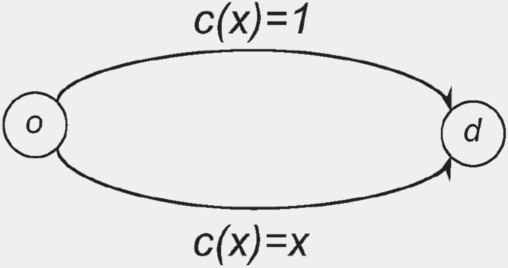


- using $v \to w$ does not lead to improvement
- $\implies$ optimal travel time is $\frac{3}{2}$ hours for each driver
- the ratio of *anarchy*: $2/\frac{3}{2} = \frac{4}{3}$

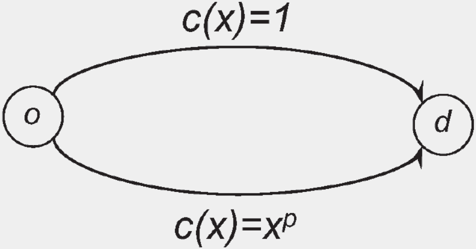
Price of Anarchy (POA)

The price of anarchy of the selfish routing network is the ratio between the equilibrium and the minumum possible average travel time.

Even smaller example: Pigou



- POA?
- equilibrium time:
  - lower route is always better
- optimal average time:
  - splitting the traffic equally between the routes



- POA?
- equilibrium time:
  - lower route is always better
- optimal average time:
  - with growing *p*, drivers on lower edge arrive instantaneously, if $(1-\varepsilon)$ where $\varepsilon > 0$ large enough is assigned

Question: *When is the POA close to 1?*

Informal answer: When there are non-linear travel times

Our model:

- $G = (V, E)$ ... directed graph
  - o ... origin
  - d ... destination
- r units of traffic
- $c_e : \mathbb{R}_+ \to \mathbb{R}_+$ ... cost function of edge e
  - monotone and non-decreasing
  - e.g. $c_e(x) = x^p$ ... travel time of x on edge e is x^p

Tight POA Bounds for Selfish Routing

Among all networks with cost functions in a set $\mathcal{C}$, the largest POA is achieved in a Pigou-like network.

Worst-case POA

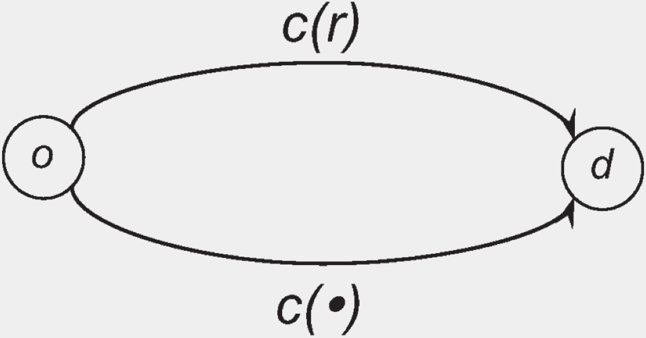
Tight POA Bounds for Selfish Routing

Among all networks with cost functions in a set $\mathcal{C}$, the largest POA is achieved in a Pigou-like network.

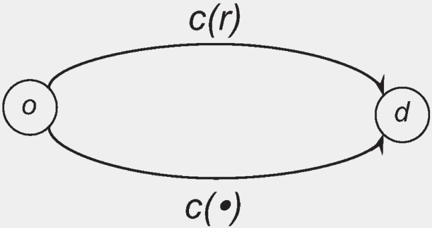
- worst-case examples are always simple
  - complexity of the network does not cause high POA
- for specific $\mathcal{C}$, we can compute worst-case POA

Description	Typical Representative	Price of Anarchy
Linear	$ax+b$	$4/3$
Quadratic	ax^2+bx+c	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.6$
Cubic	ax^3+bx^2+cx+d	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.9$
Quartic	$ax^4+bx^3+cx^2+dx+e$	$\frac{5\sqrt[4]{5}}{5\sqrt[4]{5}-4} \approx 2.2$
Degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$\frac{(p+1)\sqrt[p]{p+1}}{(p+1)\sqrt[p]{p+1}-p} \approx \frac{p}{\ln p}$

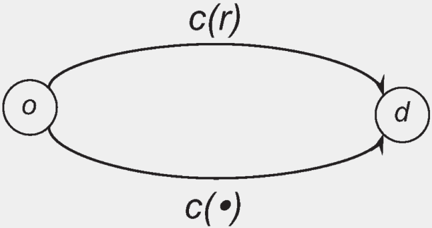
1. Two vertices: o, d
   - o ... origin, d ... destination
2. Two edges: *upper*, *lower*
3. A traffic rate $r \geq 0$
4. A cost function $c(\bullet)$ on the lower edge
5. A cost function $c(r)$ on the upper edge



The POA of a Pigou-like network



- POA?
- equilibrium time:
  - lower route is always better
- optimal time:
  - min-possible total travel time: $\inf_{r \geq x \geq 0} \{x \cdot c(x) + (r - x)c(r)\}$
  - $c(\bullet)$ is non-decreasing



- POA: $\sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$
  - $\mathcal{P}(c, r)$ ... Pigou-like network
- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$
- the lower bound on POA of networks with $c \in \mathcal{C}$

Tight POA bounds for selfish routing

- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$

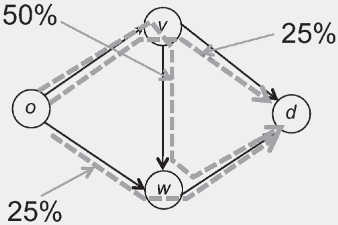
Tight POA bounds for selfish routing

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

Proof:

- $G = (V, E)$ ... selfish routing network
  - o ... origin, d ... destination
- r ... amount of traffic
- $\mathcal{P}$ ... set of o-d paths
- $\{f_P\}_{P \in \mathcal{P}}$ ... *flow*
  - f_P ... *flow on path P*
  - $\sum_{P \in \mathcal{P}} f_P = r$
  - $f_e = \sum_{P \in \mathcal{P} : e \in P} f_P$ ...*flow on edge e*

Tight POA bounds for selfish routing

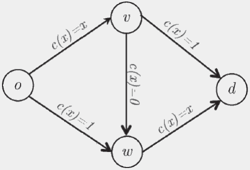


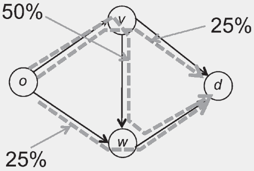
- 1 amount of traffic
- $\mathcal{P} = \{(o, v, d), (o, w, d), (o, v, w, d)\}$
  - $f_{(o,v,d)} = \frac{1}{4}$
  - $f_{(o,w,d)} = \frac{1}{4}$
  - $f_{(o,v,w,d)} = \frac{1}{2}$
- $E = \{(o, v), (v, d), (o, w), (w, d), (v, w)\}$
  - $f_{(o,v)} = f_{(w,d)} = \frac{3}{4}$
  - $f_{(o,w)} = f_{(v,d)} = \frac{1}{4}$
  - $f_{(v,w)} = \frac{1}{2}$

Equilibrium flow = *"shortest path flow"*

A flow f is an *equilibrium* if $f_{\hat{P}} > 0$ only when

$$\hat{P} \in \arg\min_{P \in \mathcal{P}} \left\{ \sum_{e \in P} c_e(f_e) \right\}.$$





- Not an equilibrium flow
  - $o \to v \to w \to d$ ... the only shortest path
- $C(f)$ ... total travel time in a flow f
  - $C(f) = \sum_{e \in E} f_e \cdot c_e(f_e) = \sum_{P \in \mathcal{P}} f_P \cdot c_P(f)$

Tight POA bounds for selfish routing

- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$

Tight POA bounds for selfish routing

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

Proof:

- $G = (V, E)$ ... selfish network, r ... traffic rate, $c \in \mathcal{C}$
- f ... equilibrium flow, f^* ... optimal flow
- $POA = \frac{C(f)}{C(f^*)} = \frac{\sum_{e \in E} f_e \cdot c_e(f_e)}{\sum_{e \in E} f_e^* \cdot c_e(f_e^*)} \geq 1$
- $\sum_{e \in E} f_e \cdot c_e(f_e) \geq \sum_{e \in E} f_e^* \cdot c_e(f_e^*)$
- we investigate the relation between $f_e \cdot c_e(f_e)$ and $f_e^* \cdot c_e(f_e^*)$

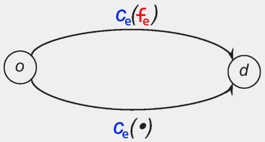
- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$

Tight POA bounds for selfish routing

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

Proof:

- we investigate the relation between $f_e \cdot c_e(f_e)$ and $f_e^* \cdot c_e(f_e^*)$



- traffic rate: f_e
- $\alpha(\mathcal{C}) \geq \frac{f_e \cdot c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e)}$

Tight POA bounds for selfish routing

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

Proof:

- $\alpha(\mathcal{C}) \geq \frac{f_e \cdot c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e)}$
- $\alpha(\mathcal{C})(f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e)) \geq f_e \cdot c_e(f_e)$
- sum over all $e \in E$
- $\alpha(\mathcal{C})(\sum_{e \in E} f_e^* \cdot c_e(f_e^*) + \sum_{e \in E}(f_e - f_e^*) \cdot c_e(f_e) \geq \sum_{e \in E} f_e \cdot c_e(f_e)$
  - $\sum_{e \in E}(f_e - f_e^*) \cdot c_e(f_e) \leq 0$ ... we show this later
  - $\alpha(\mathcal{C}) \geq 1$
- $\alpha(\mathcal{C}) C(f^*) \geq C(f)$
- $\alpha(\mathcal{C}) \geq \frac{C(f)}{C(f^*)}$

Tight POA bounds for selfish routing

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

Proof:

- why $\sum_{e \in E}(f_e - f_e^*) \cdot c_e(f_e) \leq 0$?
  - or $\sum_{e \in E} f_e \cdot c_e(f_e) \leq \sum_{e \in E} f_e^* \cdot c_e(f_e)$?

- $\sum_{e \in E} f_e c_e(f_e)$
- $\sum_{e \in E} f_e^* c_e(f_e)$
  - fixed capacities $c_e(f_e)$
  - $c_{\hat{P}}(f) = L$ for every $\hat{P}$ from equilibrium flow
  - $c_P(f) \geq L$ for every P
  - $\sum_{P \in \mathcal{P}} f_P = \sum_{P \in \mathcal{P}} f_P^* = r$

Selfish routing and POA

The price of anarchy (POA) of a selfish routing network is the ratio between the total travel time in an equilibrium flow and the minimum-possible total travel time. The POA of a selfish routing network is large only if it has *highly nonlinear* cost function.