Cooperative game theory

Martin Černý

kam.mff.cuni.cz/∼cerny
cerny@kam.mff.cuni.cz

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Selfish routing and POA

Motivation

Goal: *Construction of traffic network resistant to selfish drivers*

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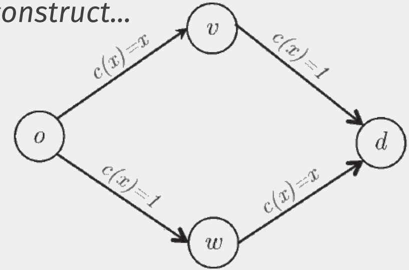
Question: *If we construct...*



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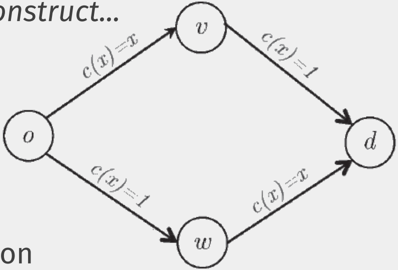

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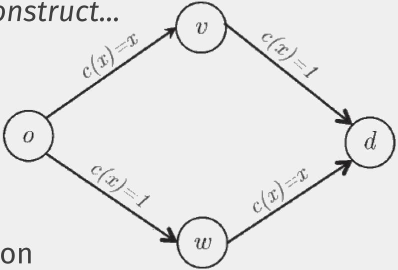

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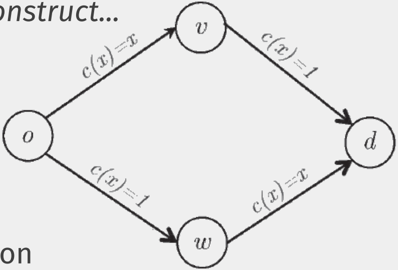

- o ... origin
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- $o \to v \to d$, $o \to w \to d$ ... routes
- wide road
  - $c(x) = 1$ ... travel time is always 1 hour
- narrow road
  - $c(x) = x$ ... travel time is proportional to the amount traffic

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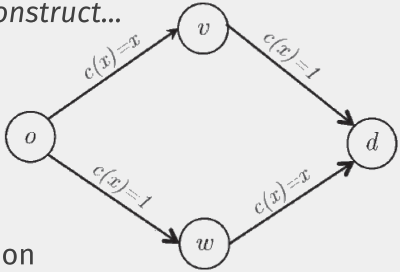

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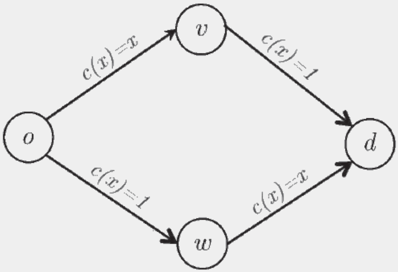
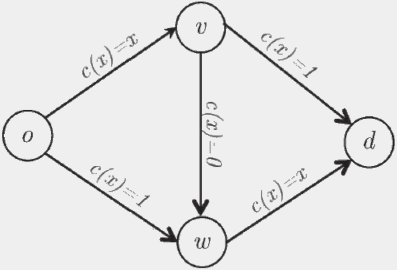
Question: *What amount of traffic to expect on each route?*



- both routes are identical $\implies$ traffic splits equally
- **Answer**: Each driver's travel time is $1 + \frac{1}{2} = \frac{3}{2}$ hours

If we wanted to improve...

Question: *What amount of traffic to expect on each route?*



- new route $o \to v \to w \to d$

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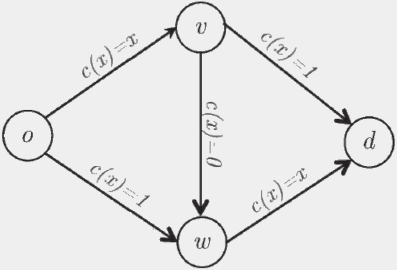

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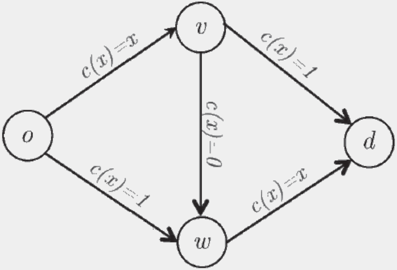

- new route $o \to v \to w \to d$
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- $\implies$ all drivers prefer this route

...WE WOULD MAKE IT WORSE!

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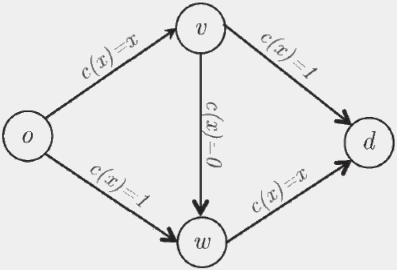
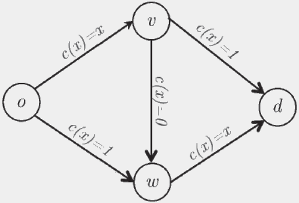


- new route $o \to v \to w \to d$
- it is always better than $o \to v \to d$ or $o \to w \to d$
- $\implies$ all drivers prefer this route
- $\implies$ **each driver drives** $1 + 1 = 2$ **hours!**

Question: *How much worse is this?*

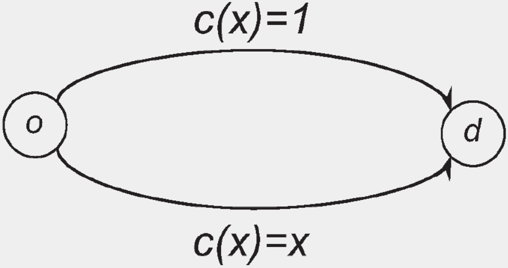


- using $v \to w$ does not lead to improvement
- $\implies$ optimal travel time is $\frac{3}{2}$ hours for each driver
- the ratio of *anarchy*: $2/\frac{3}{2} = \frac{4}{3}$

Price of Anarchy (POA)

The price of anarchy of the selfish routing network is the ratio between the equilibrium and the minumum possible average travel time.

Even smaller example: Pigou



- POA?

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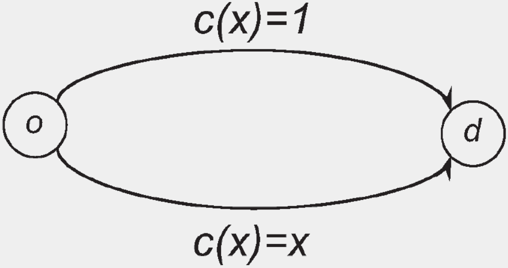

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- equilibrium time:
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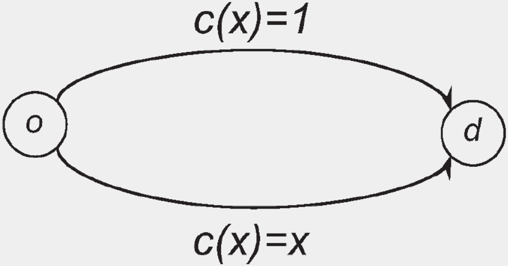

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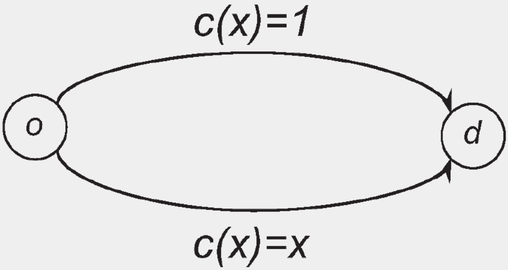

- POA?
- equilibrium time: 1
  - lower route is always better
- optimal average time:

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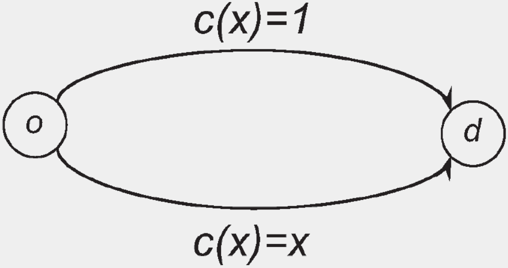

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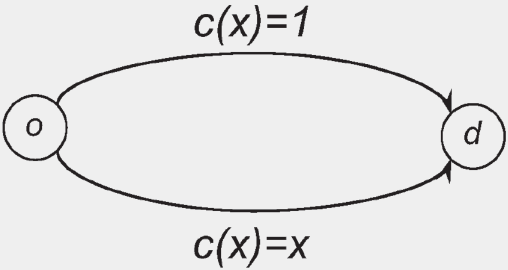
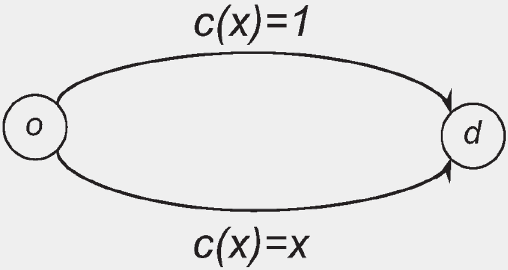
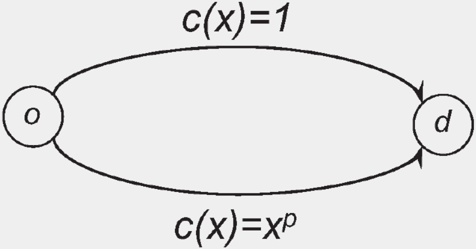


- POA?
- equilibrium time: 1
  - lower route is always better
- optimal average time: $\frac{3}{4}$
  - splitting the traffic equally between the routes

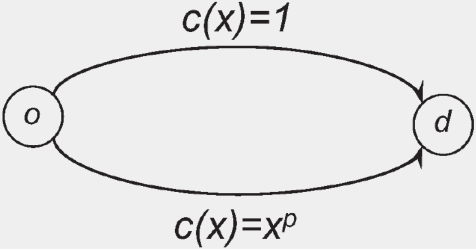


- POA? $\frac{4}{3}$
- equilibrium time: 1
  - lower route is always better
- optimal average time: $\frac{3}{4}$
  - splitting the traffic equally between the routes



- POA?

Pigou - nonlinear variant



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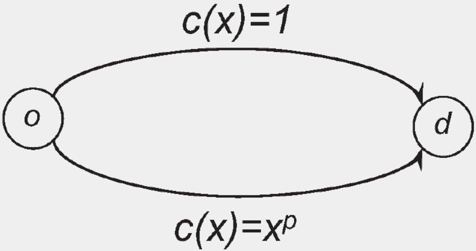

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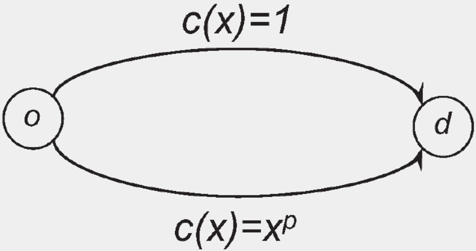

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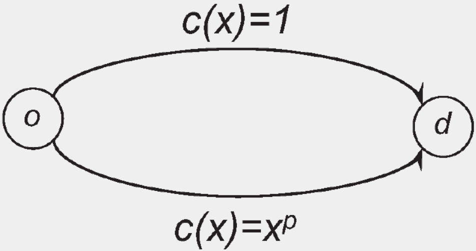

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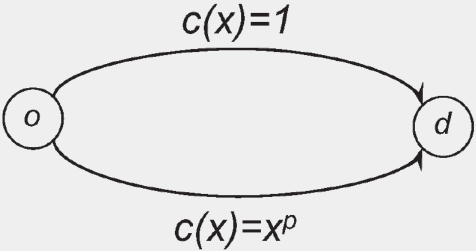

- POA?
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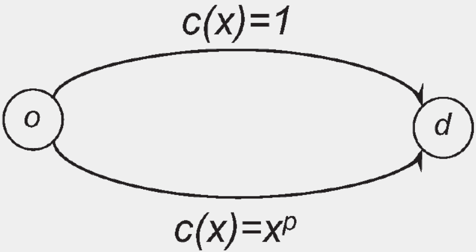

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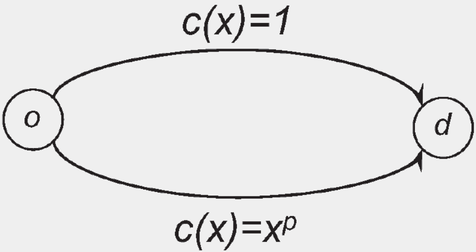
PIGOU - NONLINEAR VARIANT



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Question: *When is the POA close to 1?*

Informal answer: When there are non-linear travel times

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Our model:

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- $G = (V, E)$ ... directed graph
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- $c_e : \mathbb{R}_+ \to \mathbb{R}_+$ ... cost function of edge e
  - monotone and non-decreasing
  - e.g. $c_e(x) = x^p$ ... travel time of x on edge e is x^p

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Among all networks with cost functions in a set $\mathcal{C}$, the largest POA is achieved in a Pigou-like network.

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Worst-case POA

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  - complexity of the network does not cause high POA

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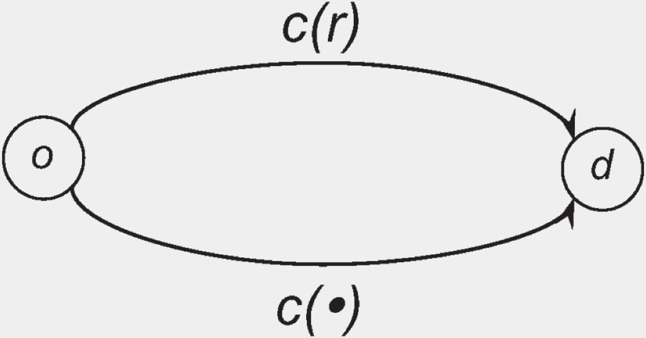
- worst-case examples are always simple
  - complexity of the network does not cause high POA
- for specific $\mathcal{C}$, we can compute worst-case POA

Description	Typical Representative	Price of Anarchy
Linear	$ax + b$	$4/3$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.6$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.9$
Quartic	$ax^4 + bx^3 + cx^2 + dx + e$	$\frac{5\sqrt[4]{5}}{5\sqrt[4]{5}-4} \approx 2.2$
Degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$\frac{(p+1)\sqrt[p]{p+1}}{(p+1)\sqrt[p]{p+1}-p} \approx \frac{p}{\ln p}$

Pigou-like network

1. Two vertices: o, d
   - o ... origin, d ... destination
2. Two edges: *upper*, *lower*
3. A traffic rate $r \geq 0$
4. A cost function $c(\bullet)$ on the lower edge
5. A cost function $c(r)$ on the upper edge



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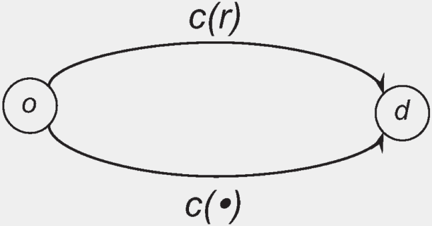

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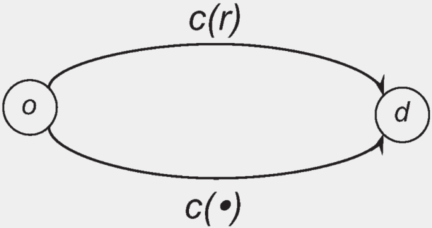

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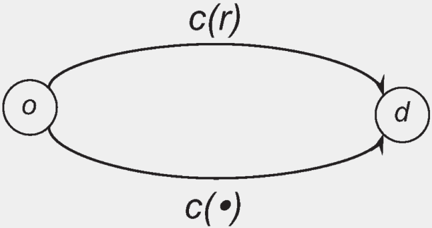

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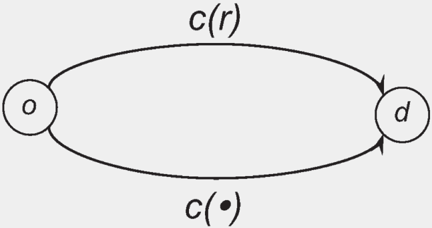

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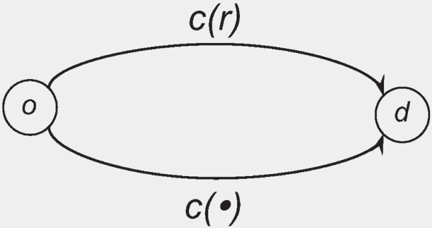

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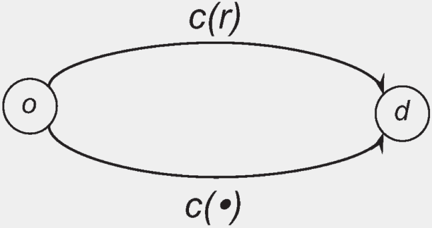

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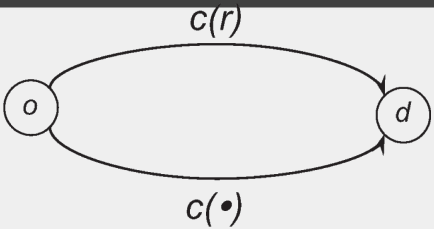

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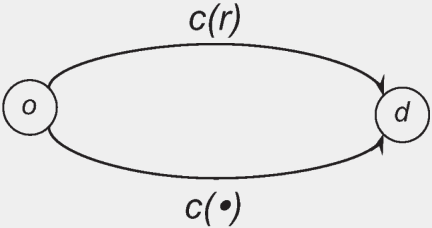

- POA: $\sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right\}$
  - $\mathcal{P}(c, r)$ ... Pigou-like network
- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right\}$
- the lower bound on POA of networks with $c \in \mathcal{C}$

Tight POA bounds for selfish routing

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For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

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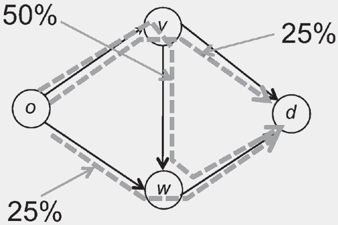
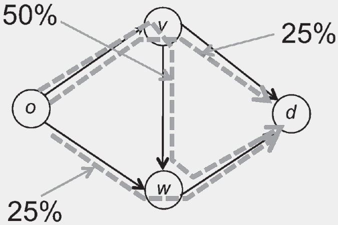
  - $\sum_{P \in \mathcal{P}} f_P = r$
  - $f_e = \sum_{P \in \mathcal{P}: e \in P} f_P$ ...*flow on edge e*

Tight POA bounds for selfish routing



- 1 amount of traffic



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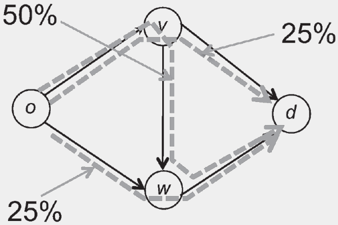

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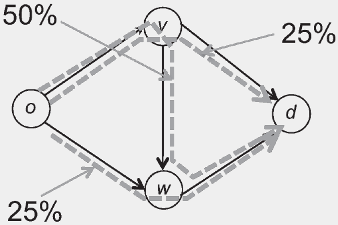

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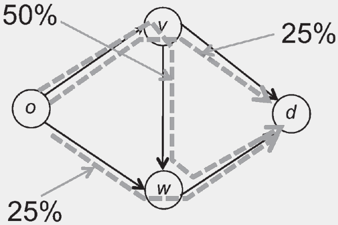

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A flow f is an *equilibrium* if $f_{\hat{P}} > 0$ only when

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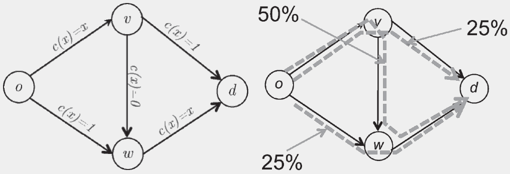

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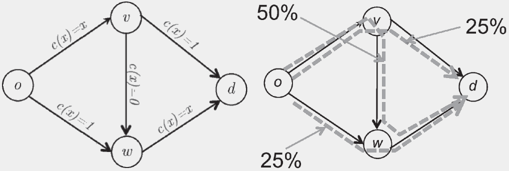

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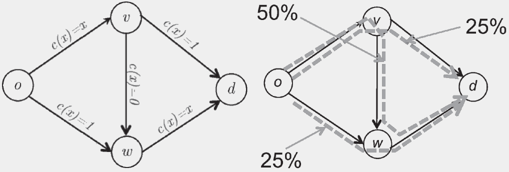

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- $C(f)$ ... total travel time in a flow f
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Tight POA bounds for selfish routing

- Pigou bound: $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right\}$

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For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.

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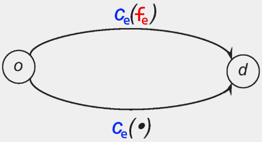

- traffic rate: f_e
- $\alpha(\mathcal{C}) \geq \frac{f_e \cdot c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e)}$

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- $\sum_{e \in E} f_e c_e(f_e) = \sum_{P \in \mathcal{P}} f_P \cdot c_P(f) = r \cdot L$
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  - $c_{\hat{P}}(f) = L$ for every $\hat{P}$ from equilibrium flow
  - $c_P(f) \geq L$ for every P
  - $\sum_{P \in \mathcal{P}} f_P = \sum_{P \in \mathcal{P}} f_P^* = r$

Selfish routing and POA

The price of anarchy (POA) of a selfish routing network is the ratio between the total travel time in an equilibrium flow and the minimum-possible total travel time. The POA of a selfish routing network is large only if it has *highly nonlinear* cost function.