Cooperative game theory

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- Teaching: Martin Černý, Martin Loebl
  - Introduction to cooperative games (8)
  - Concepts of fairness (1-2)
  - Routing games (1-2)
  - Price of anarchy (1-2)

Materials

- https://kam.mff.cuni.cz/~cerny/teach/22-23/coop.html

Examination

- two smaller exams
  - possible in the mids of semester

Questions?

Motivations and introduction

Goal: *To analyse and solve conflicting interaction between subjects*

Basic model: game in **normal form** $(N, \mathcal{S}, v)$:

- N ... set of n players
- $\mathcal{S} = S_1 \times S_2 \times \cdots \times S_n$ ... strategy profiles
  - S_i ... strategies of player i
- $v \colon \mathcal{S} \to \mathbb{R}^n$ ... utility function
  - $v(S)_i$ ... utility of player i under strategy S

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Basic model: game in **normal form** $(N, \mathcal{S}, v)$:

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Fundamental question: *What strategy should player i choose?*

Nash equilibrium

Idea: Deviation from the actual strategy to a new strategy does not improve the outcome.

Nash equilibrium

Strategy profile $(s_1, \ldots, s_n)$ is **Nash equilibrium**, if it holds for every player i,

$$v_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq v_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$$

for every $t_i \in S_i$.

- Is this model suitable for the analysis of *cooperation*?

Prisoner A \ Prisoner B	Silence	Speak
Silence	-1/-1	-3/0
Speak	0/-3	-2/-2

- Nash equilibrium does not lead to cooperation
- Main problem?
  - **The prisoners cannot agree on a strategy in advance**
- Solution?
  1. Fix the existing model
     - $\Longrightarrow$ Pareto optimum
     - $\Longrightarrow$ Nash "cooperative" solution
  2. Create a new model
     - model of *cooperative games*

Goal of model? *Understanding of the behaviour of markets*

Market

A **market** $(N, \mathbb{R}^m_+, A, w)$ is given by:

- N ... set of players
- $\mathbb{R}^m_+$ ... comodity space
- $A \in \mathbb{R}^{m \times n}$ ... matrix of initial allocations
  - $A = \begin{pmatrix} | & | & & | \\ a^1 & a^2 & \dots & a^n \\ | & | & & | \end{pmatrix}$
  - $a^i \in \mathbb{R}^m$ ... commodities owned by player i
- $w = w_1 \times w_2 \times \dots w_n$ ... utility function
  - $w_i \colon \mathbb{R}^m_+ \to \mathbb{R}$ ... utility function of player i
  - w_i is continuous, concave function

Market games are **transferable utility** games

- better understanding after we define cooperative games
- meanwhile: *money* used for evaluation of profit
- formally: $W_i(x, \zeta) = w_i(x) + \zeta$
    - $x \in \mathbb{R}^m$ ... vector of commodities
    - $\zeta \in \mathbb{R}$ ... money
        - ζ may be negative
        - amount we paid to buy the commodities

Goal of market games? Determine *trade* on the market

Trade of coalition S

Trade for market $(N, \mathbb{R}^n_+, A, w)$ between players in $\emptyset \neq S \subseteq N$ is a collection $(x^i, \zeta^i)_{i \in S}$ satisfying:

1. $\sum_{i \in S} x^i = \sum_{i \in S} a^i$ (preservation of commodities)
2. $\sum_{i \in S} \zeta^i = 0$ (preservation of money).

Questions:

- Which coalitions agree on trading?
- What trade will occur within a given coalition?
  - *What is the profit of player* ***i*** *for a given market?*

Goal: *Understanding of voting systems a coalition formation*

Voting game

A **voting game** $(N, \mathcal{W})$ is given by:

- N ... set of player
- $\mathcal{W} \subseteq 2^N$ ... množina vítězných koalic
  - $\emptyset \notin \mathcal{W}$
  - $S \subseteq T \wedge S \in \mathcal{W} \implies T \in \mathcal{W}$

Special case: **(Weighted) majority games**

- $v_i \in \mathbb{R}$ ... power of player i
- $v(S) = \sum_{i \in S} v_i$ ... power of coalition $S \subseteq N$
- $S \in \mathcal{W} \iff v(S) \geq k$
  - $k \in \mathbb{R}$ ... voting quota

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Application: Voting system UN Security Council

- 15 members (5 permanent)
  - permanent members: right of veto
- to adopt resolution: 9 votes needed
- $N = \{1, \ldots, 15\}$
- $k = 39$
- $v_i = \begin{cases} 7 & i = 1, \ldots, 5 \\ 1 & i = 6, \ldots, 15 \end{cases}$

Cooperation - examples of models: *Cost-sharing problem*

Goal: *Does cooperation lead to reduction of costs?*

- N ... set of players
- $c(S)$ ... value of costs for a service shared by $S \subseteq N$
  - $c(S) + c(T) \geq c(S \cup T)$
  - a common solution simulated by union of individual solutions

Examples:

- drinking water supply
- municipal waste collection
- subscriptions to scientific journals

Cooperation - examples of models: *Minimal spanning-tree games*

Goal: *Find the best connection of players to a source*

- $N = N' \cup \{0\}$ ... set of players + source
- c_{ij} ... cost of connecting i, j
- solution: a network, where each $i \in N$ is connected to 0 with minimal sum of costs

Cooperative game

A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

- Γ^n ... set of n-person cooperative games
- $S \subseteq N$ ... coalition
- $v(S)$ ... values of coalition
- usually $N = \{1, \dots, n\}$
  - (S, v_S) is **subgame** (N, v):
    - $v_S\colon 2^S \to \mathbb{R}$
    - $v_S(T) := v(T)$ pro $T \subseteq S$

- $(N, \mathbb{R}^m_+, A, w)$ ... market

Feasible S-allocation

Feasible S**-allocations** is $(a^i_S)_{i \in S}$ satisfying

$$\sum_{i \in S} a^i_S = \sum_{i \in S} a^i.$$

We denote the set of feasible S-allocations by $\mathcal{A}_S$.

- $(N, \mathbb{R}^m_+, a, w)$ ... market

Feasible S-allocation

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We denote the set of feasible S-allocations by $\mathcal{A}_S$.

Market game

Cooperative game (N, v) is **market game**, if there is market $(N, \mathbb{R}^m_+, A, w)$ satisfying

$$v(S) = \max\{\sum_{i \in S} w^i(a^i_S) \mid (a^i_S)_{i \in S} \in \mathcal{A}_S\}.$$

Voting game

A **voting game** $(N, \mathcal{W})$ is given by:

- N ... set of players
- $\mathcal{W} \subseteq 2^N$ ... set of winning coalitions
  - $\emptyset \notin \mathcal{W}$
  - $S \subseteq T \land S \in \mathcal{W} \implies T \in \mathcal{W}$

Simple game

Simple game (N, v) satisfies:

1. $v\colon 2^N \to \{0, 1\}$
2. $S \subseteq T \land v(S) = 1 \implies v(T) = 1$

- $v(S) = 1 \iff S \in \mathcal{W}$

Cooperation - examples of models: *Cost-sharing problem*

Goal: *Does cooperation lead to reduction of costs?*

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Cooperative game

- $c\colon 2^N \to \mathbb{R}$
- a so called *cost game* (N, c)
  - $c(S)$ ... values represent *costs*
  - $v(S)$ ... values represent *profit*

Example: *Bigger coalitions are better coalitions...*

- **monotonic game** ($S \subseteq T \subseteq N$)

$$v(S) \leq v(T)$$

- **superadditive game** ($S, T \subseteq N, S \cap T = \emptyset$)

$$v(S) + v(T) \leq v(S \cup T)$$

  - $S \cup T$ can behave as separate S and T
  - does not hold if there are costs of managing coalitions
  - cost-games: $c(S) + c(T) \geq v(S \cup T)$ ... **subadditive game**

- **convex game** ($S, T \subseteq N$)

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$$

Example: *Bigger coalitions are better coalitions...*

- **monotonic game** ($S \subseteq T \subseteq N$)

$$v(S) \leq v(T)$$

- **superadditive game** ($S, T \subseteq N, S \cap T = \emptyset$)

$$v(S) + v(T) \leq v(S \cup T)$$

- **convex game** ($S, T \subseteq N$)

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$$

  - also **supermodular game**
  - cost-game: $c(S) + c(T) \geq c(S \cap T) + c(S \cup T)$
    - **concave (submodular)**

Money first!

- **Payoff vector $\mathbf{x} \in \mathbb{R}^n$**
  - x_i represents payoff of player i
- Vector $\mathbf{x} \in \mathbb{R}^n$ is **efficient**, if $\sum_{i \in N} x_i = v(N)$
  - Usually, we distribute $v(N)$
    1. *value of cooperation* $v(N)$
    2. *shared costs* $c(N)$
- Vector $\mathbf{x} \in \mathbb{R}^n$ is **individually rational**, if $x_i \geq v(i)$
  - players prefer x_i over $v(i)$

- $\mathcal{I}^*(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$ ... **preimputation**
  - $x(S) := \sum_{i \in S} x_i$
- $\mathcal{I}(v) = \{x \in \mathcal{I}^*(v) \mid \forall i \in N : x_i \geq v(i)\}$ ... **imputation**

Solution concepts

- Set of payoff vectors satisfying further properties are **solution concepts**
- Can reflect payoff distribution, which is
  - *...fair...*
  - *...non-discriminatory...*
  - *...stable (players will accept it)...*
  - ...

Examples of solution concepts: The core

Idea: *Payoff distribution leads to cooperation...*

The core

For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\}.$$

- assumption: *homo economicus*
  - model of human as a player
  - strictly rational and selfish
  - follows his subjective goals
- $v(N)$ ... value, which is distributed among players
- $x(S) > v(S) \implies$ coalition S does not leave N
  - would lead to (S, v_S)
  - $v(S)$ ... distributed value

Examples of solution concepts: The Shapley value

Idea: *Divide the profit in a fair way...*

The Shapley value

For a cooperative game (N, v), the **Shapley value** $\phi(v)$ of player i is

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S \cup i) - v(S)\right)$$

- $v(S \cup i) - v(S)$
  - *marginal contribution* (of player i in $S \cup i$)
- $\frac{s!(n-s-1)!}{n!}$
  - weights reflecting different sizes of coalitions
- $\sum_{S \subseteq N \setminus i}$
  - sum of all marginal contributions of i

Solution concepts

Formally:

1. sets of payoff vectors
   - $\Sigma(v) = \{x \in \mathbb{R}^n \mid \dots\}$
2. functions on games
   - $\Sigma\colon \Gamma^n \to 2^{\mathbb{R}^n}$

Formally:

1. sets of payoff vectors
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We distringush

1. **single-point** solution concepts
   - as a set: $\Sigma(v) = \{x\}$
     - we prefer: $\Sigma(v) = x$
   - as a function: $\Sigma\colon \Gamma^n \to \mathbb{R}$
2. **multi-point** solution concepts

Alternative way to define the Shapley value

It is possible to define it using its properties...

The Shapley value

The **Shapley value** $\phi(v)$ is the only function $f\colon \Gamma^n \to \mathbb{R}$ satisfying for all games $(N, v), (N, w)$:

1. (AXIOM OF EFFICIENCE)
   - $\sum_{i \in N} f_i(v) = v(N)$
2. (AXIOM OF SYMMETRY)
   - $\forall i, j \in N \; (\forall S \subseteq N \setminus \{i, j\} : v(S \cup i) = v(S \cup j)) \implies f_i(v) = f_j(v)$
3. (AXIOM OF NULL PLAYER)
   - $\forall i \in N \; (\forall S \subseteq N : v(S) = v(S \cup i)) \implies f_i(v) = 0$
4. (AXIOM OF ADDITIVITY)
   - $v, w \in \Gamma^n : f(v + w) = f(v) + f(w)$

Cooperative games

A **cooperative game** is given by a set of players and real values representing the profit of each subset of players (*coalition*). These values are encoded by the *characteristic function* of a game. The goal of the cooperative game theory is to find payoffs (in form of **payoff vectors**) for individual players based on values of the game. Payoff vectors satisfying further properties form **solution concepts**. These can be expressed as:

1. sets of payoff vectors
2. functions on games
    - 2.1 defined by formula
    - 2.2 defined by its properties