On the Tree Search Problem with Non-uniform costs

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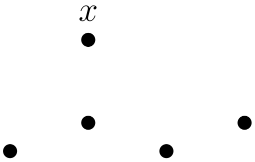
27^{th} Cumberland Conference on Combinatorics,
Graph Theory & Computing
May 17, 2014

The search problem - general version

Input: n objects where one is special
Output: The special object



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Answers: Yes/No.

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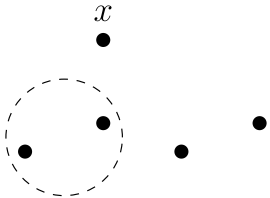

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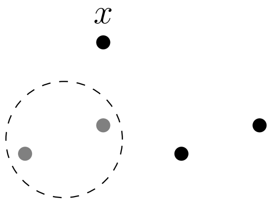

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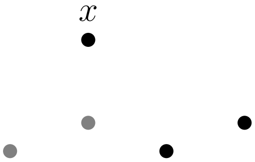

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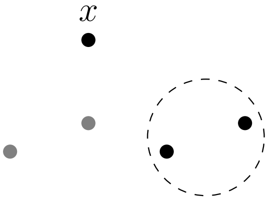

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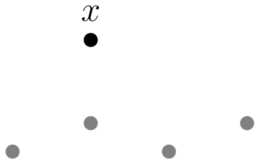

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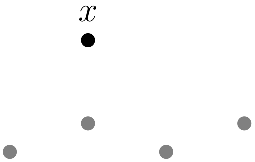

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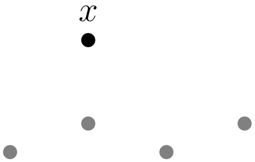

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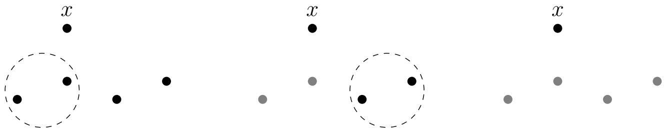
Find a treasure quickly!

Online and offline versions

Online: Questions and answers alternate.



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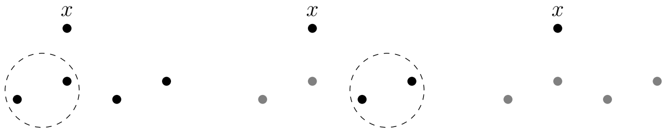

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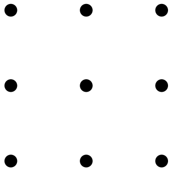

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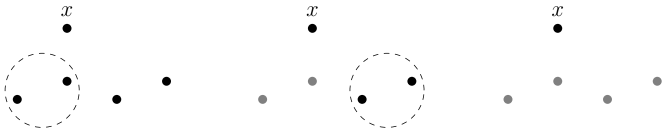

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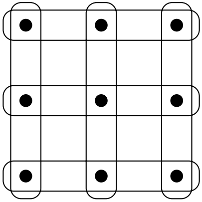

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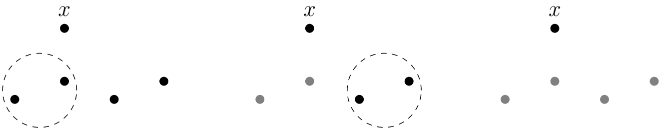

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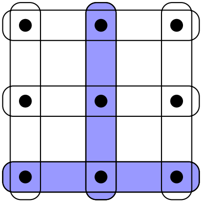

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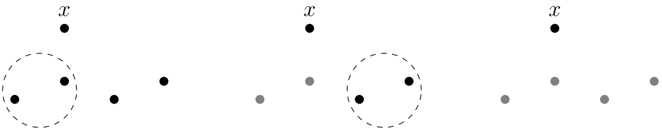

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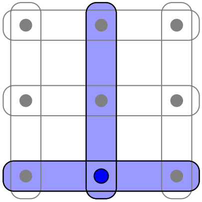

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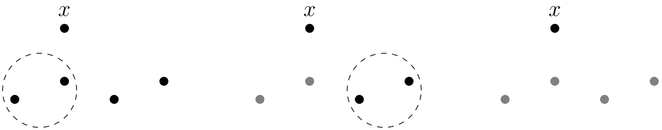

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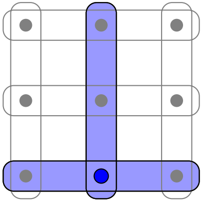

We consider only online version.

Searching

Find 4 in an ordered list using binary search.

$1 \quad 4 \quad 6 \quad 7$

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$$\boxed{1 \quad 4} \quad 6 \quad 7$$

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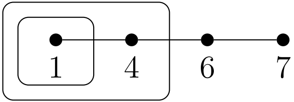

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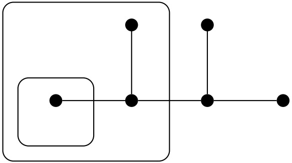
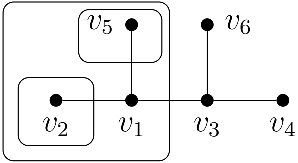


Searching in trees

Input: Tree T, one hidden vertex



Output:

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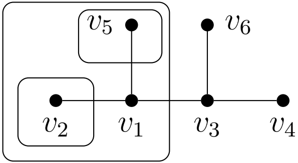

Output: Decision tree D of minimum depth

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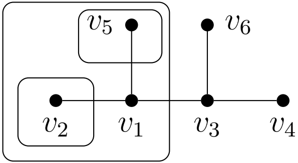

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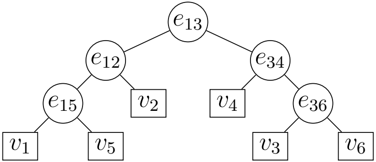

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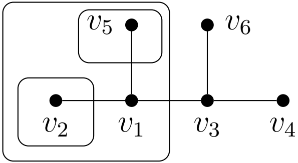

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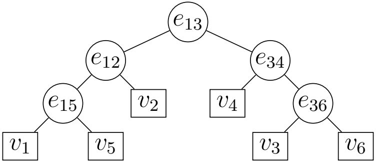
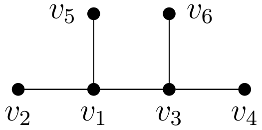


Theorem (Lam, Yue '98)

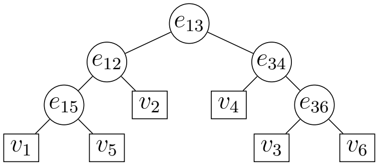
An optimal decision tree D can be computed in a polynomial time.

Searching in trees with costs on edges

Input: Tree T

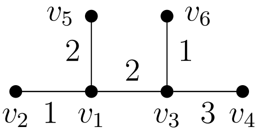


Output: Optimal decision tree D



Searching in trees with costs on edges

Input: Tree T, cost $c : E(T) \to \mathbb{Z}$



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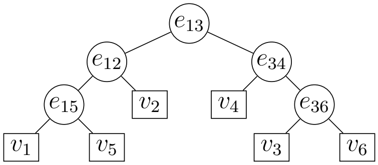

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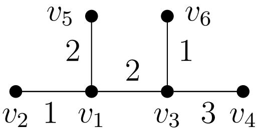

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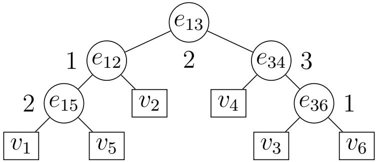

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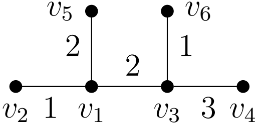
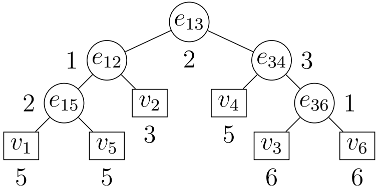


Output: Optimal decision tree D (cheapest worst case)



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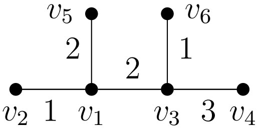

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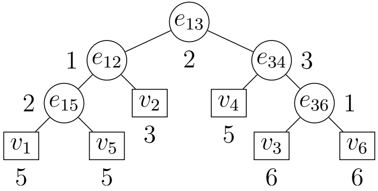

$$cost(D) := \max_{v \in V(T)} \{cost^D(v)\} \quad OPT(T, c) := \min_{D} \{cost(D)\}$$

Known results

Input: Tree T on n vertices, cost c
Output: Decision tree D with $cost(D) = OPT(T, c)$

Theorem (Dereniowsky, '06)

NP-complete if diameter of T is 10
$O(\log(n))$*-approximation algorithm*

$$cost(D) = O(\log(n)) \cdot OPT(T, c)$$

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$O\left(\frac{\log n}{\log\log\log n}\right)$*-approximation algorithm*

Our results

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Theorem (Cicalese, Keszegh, L., Pálvölgyi, Valla)

NP-complete for diameter 6 subdivided stars (from Knapsack)
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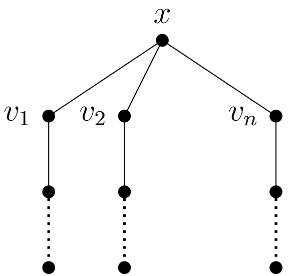
Problem
Is there $O\left(\frac{\log n}{\log n}\right)$*-approximation algorithm?*

Proof warm-up

Lemma

There is a 2-approximation algorithm for subdivided stars.



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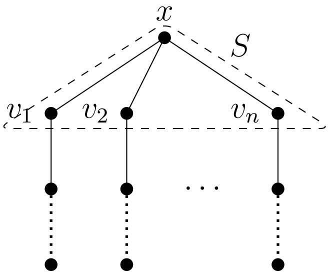

Find a decision tree for S

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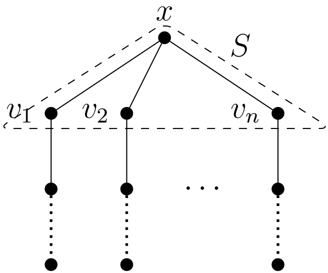

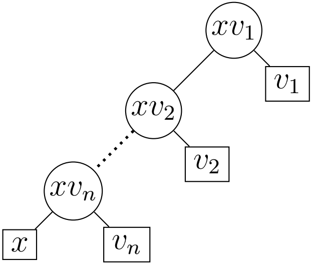

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Cost : $OPT(T, c)$

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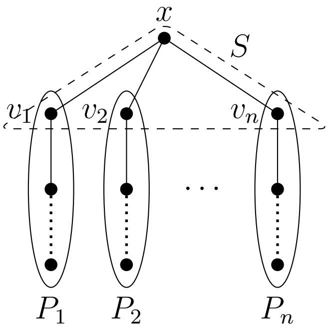

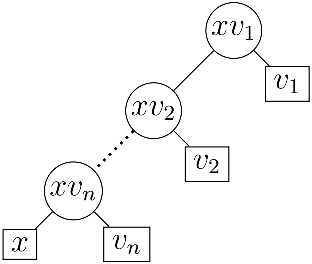

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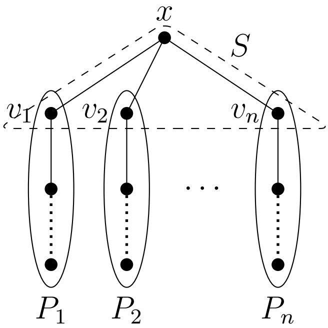

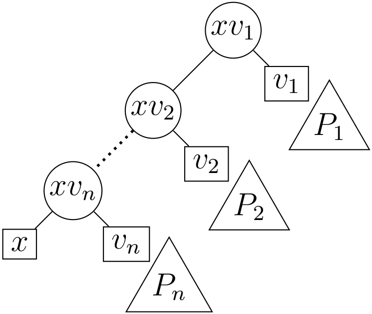

Find a decision tree for S and for P_i

$Cost : OPT(T, c)$

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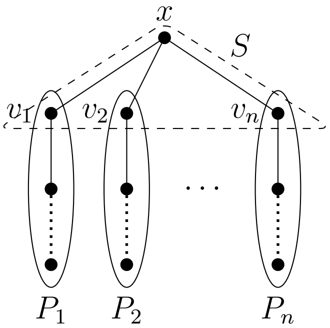
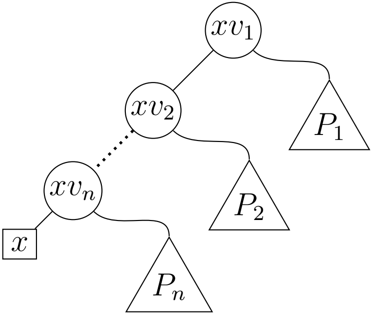
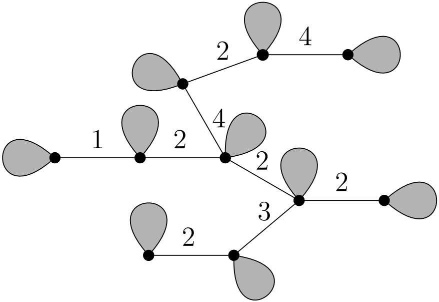




Find a decision tree for S and for P_i and combine them.
Cost : $OPT(T, c) + OPT(T, c)$

Proof sketch (algorithm for trees)

Idea: Find a small separator S, small resulting components, recurse



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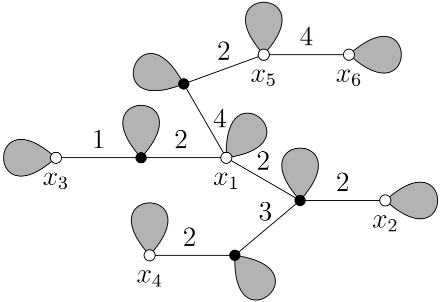

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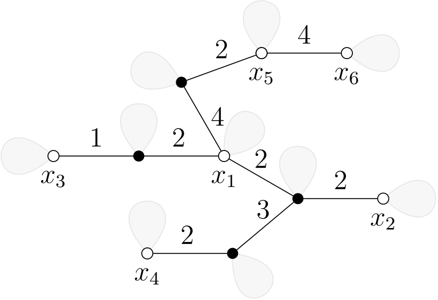

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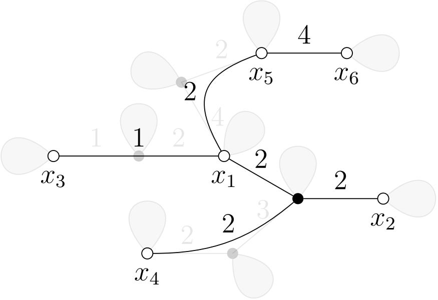

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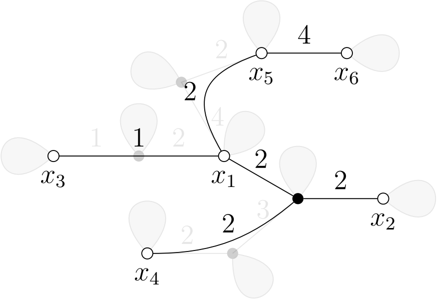
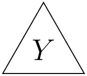


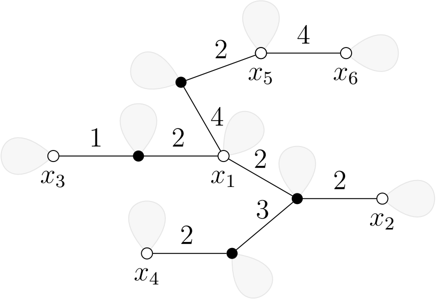


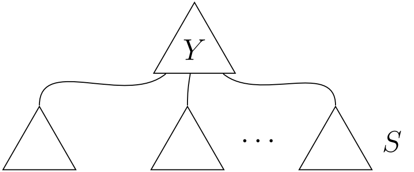
Fix $t = \log(n)$; pick t centroids $S = \{x_1 \ldots\}$; (components n/t)
build auxiliary graph Y of size $2t$, solve Y in $O(t2^{2t})$

Cost: $OPT(T, c)$

Proof sketch (algorithm for trees)

Idea: Find a small separator S, small resulting components, recurse





Fix $t = \log(n)$; pick t centroids $S = \{x_1 \ldots\}$; (components n/t)
build auxiliary graph Y of size $2t$, solve Y in $O(t2^{2t})$
solve subdivided stars in S

Cost: $OPT(T, c) + 2OPT(T, c)$

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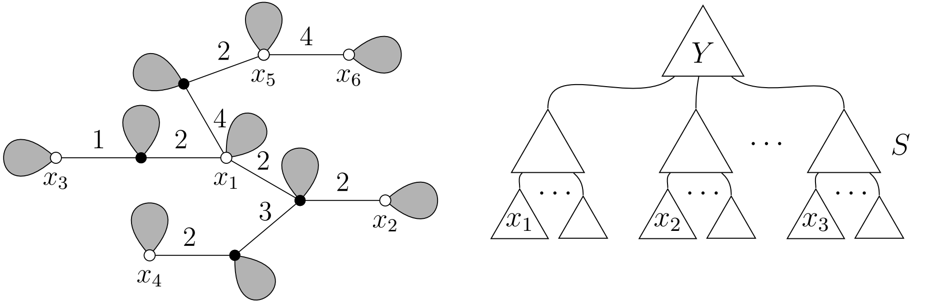

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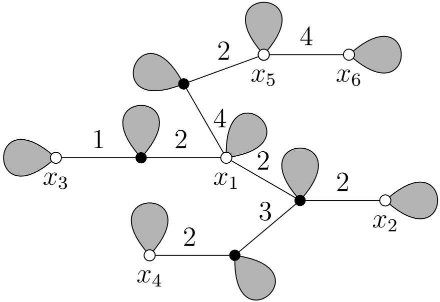

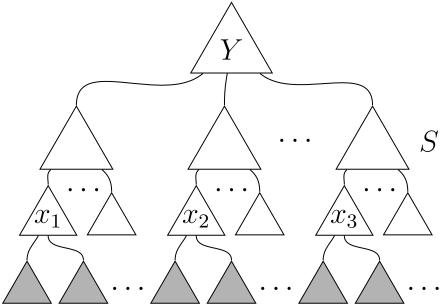

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Cost: $(OPT(T,c) + 2OPT(T,c) + OPT(T,c)) \times k$

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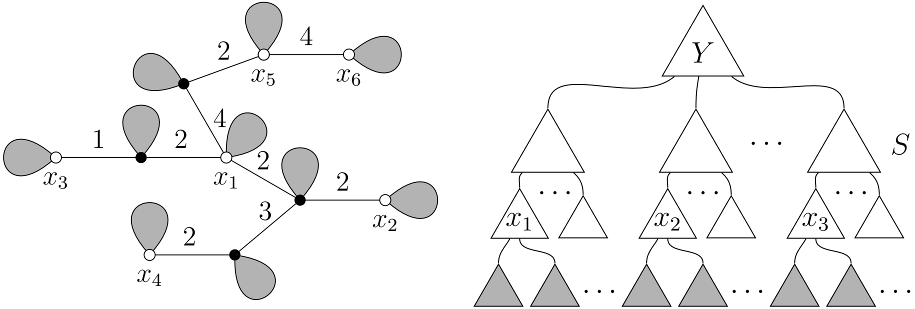

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solve subdivided stars in S, solve neighbors of S, recursion $\times k$;
$t^k = n$ hence $k = \log(n)/\log(t)$
Cost: $(OPT(T,c) + 2OPT(T,c) + OPT(T,c)) \times k$

Thank you for your attention!