Short proofs of coloring theorems on planar graphs

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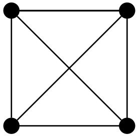
Definitions (4-critical graphs)

graph $G = (V, E)$

coloring is $\varphi : V \to C$ such that $\varphi(u) \neq \varphi(v)$ if $uv \in E$

G is a *k-colorable* if coloring with $|C| = k$ exists

G is a *4-critical graph* if G is not 3-colorable
but every $H \subset G$ is 3-colorable.



Inspiration

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

Recently reproved by Kostochka and Yancey using

Theorem (Kostochka and Yancey '12)

If G is 4-critical graph, then

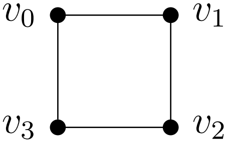
$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

used as $3|E(G)| \geq 5|V(G)| - 2$

Every planar triangle-free graph is 3-colorable.

Let G be a minimal counterexample - not 3-colorable but every subgraph is. i.e. G is 4-critical

CASE1 G contains a 4-face (try 3-color G)



CASE2 G contains no 4-faces

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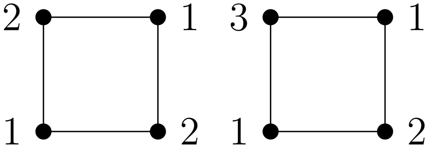

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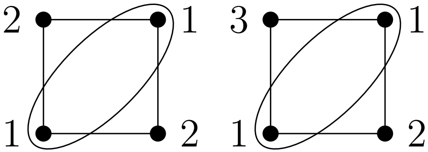

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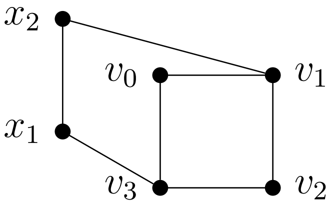

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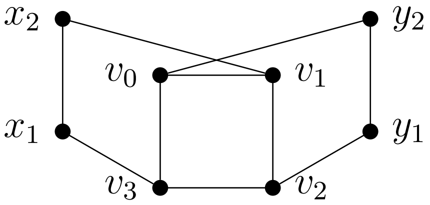

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CASE1 G contains a 4-face (try 3-color G)

CASE2 G contains no 4-faces

$|E(G)| = e, |V(G)| = v, |F(G)| = f$.

- $v - 2 + f = e$ by Euler's formula
- $2e \geq 5f$ since face is at least 5-face
- $5v - 10 + 5f = 5e$
- $5v - 10 + 2e \geq 5e$
- $5v - 10 \geq 3e$ (our case)
- $3e \geq 5v - 2$ (every 4-critical graph)

Generalizations?

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

Can be strengthened?

Generalizations?

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

Can be strengthened?

Yes! - recall that CASE2

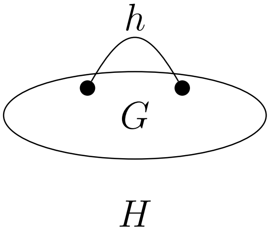
- $5v - 10 \geq 3e$ (no 3-,4-faces)
- $3e \geq 5v - 2$ (every 4-critical graph)

has some gap.

Adding a bit

Theorem (Aksenov '77; Jensen and Thomassen '00)

Let G be a triangle-free planar graph and H be a graph such that $G = H - h$ for some edge h of H. Then H is 3-colorable.



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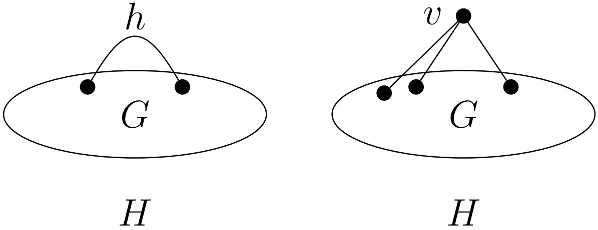
Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 3. Then H is 3-colorable.



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Theorem (Aksenov '77; Jensen and Thomassen '00)

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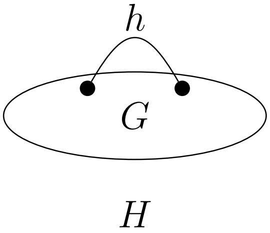
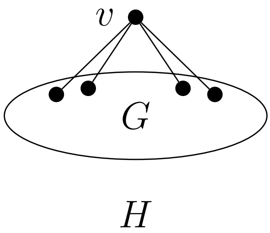
Theorem

Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 4. Then H is 3-colorable.

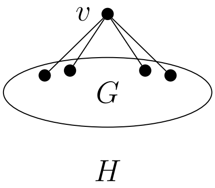




For proof

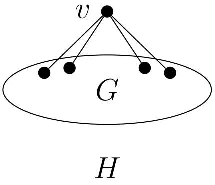
Theorem

Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 4. Then H is 3-colorable.



Proof

G plane, triangle-free, $G = H - v$,
H is 4-critical



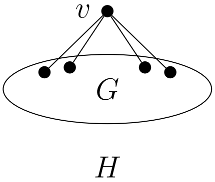
CASE1: No 4-faces in G

$V(H) = v, E(H) = e, V(G) = v - 1, E(G) = e - 4, F(G) = f$

- $5f \leq 2(e - 4)$ since G has no 4-faces
- $(n - 1) + f - (e - 4) = 2$ by Euler's formula
- $5n + 5f - 5e = -5$
- $5n - 3e - 8 \geq -5$
- $5n - 3 \geq 3e$ (our case)
- but $3e \geq 5n - 2$ (H is 4-criticality)

CASE2: 4-face $(v_0, v_1, v_2, v_3) \in G$

Proof



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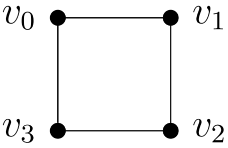

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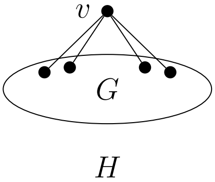

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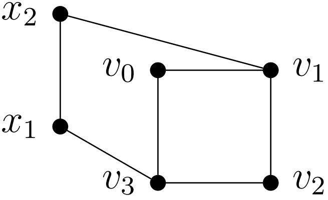

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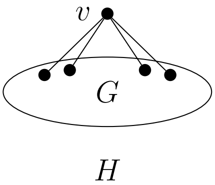

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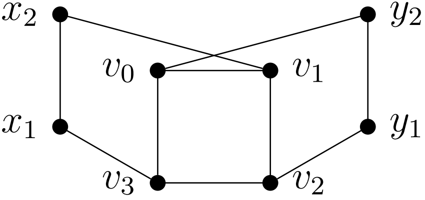
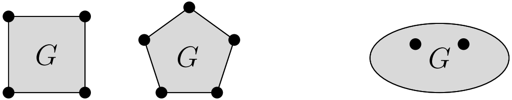


Precoloring

Theorem (Grötzsch '59)

Let G be a triangle-free planar graph and F be a face of G of length at most 5. Then each 3-coloring of F can be extended to a 3-coloring of G.



Theorem (Aksenov et al. '02)

Let G be a triangle-free planar graph. Then each coloring of two non-adjacent vertices can be extended to a 3-coloring of G.

For proof

Theorem (Grötzsch '59)

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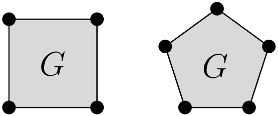
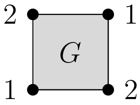


Proof

If G is triangle-free planar, F is a precolored 4-face or 5-face, then precoloring of F extends.

CASE1: F is a 4-face

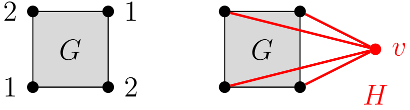


CASE2: F is a 5-face

Proof

If G is triangle-free planar, F is a precolored 4-face or 5-face, then precoloring of F extends.

CASE1: F is a 4-face H is 3-colorable



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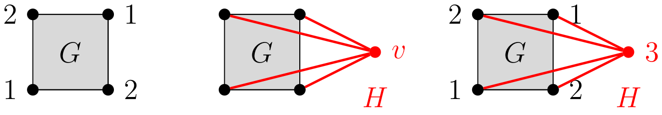

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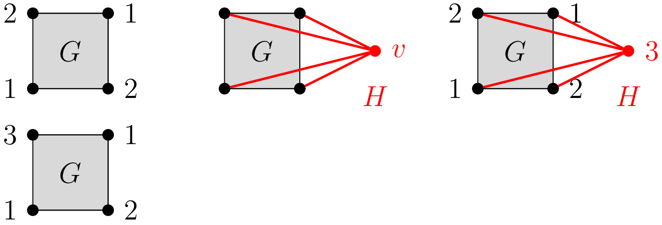

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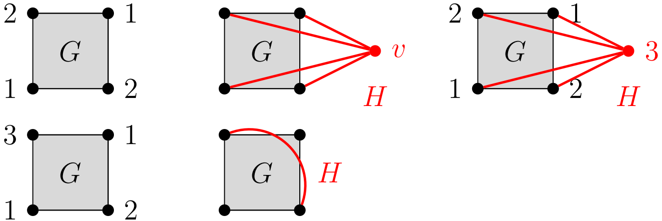

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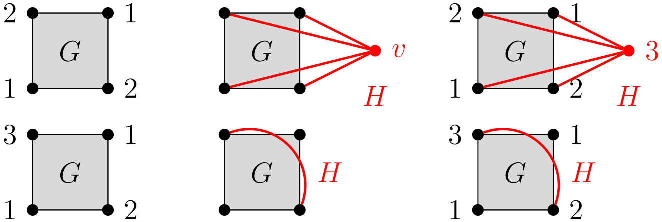

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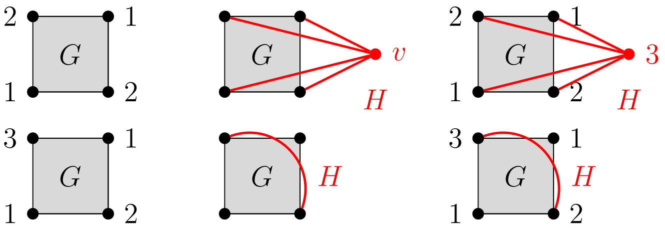

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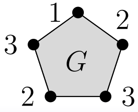
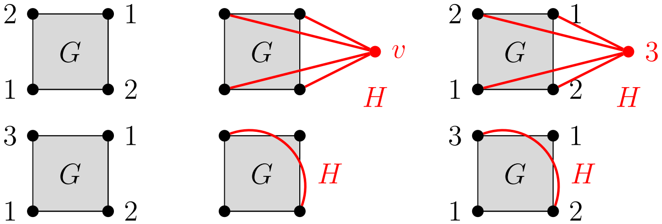


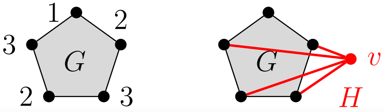
Proof

If *G* is triangle-free planar, *F* is a precolored 4-face or 5-face, then precoloring of *F* extends.

CASE1: *F* is a 4-face *H* is 3-colorable



CASE2: *F* is a 5-face



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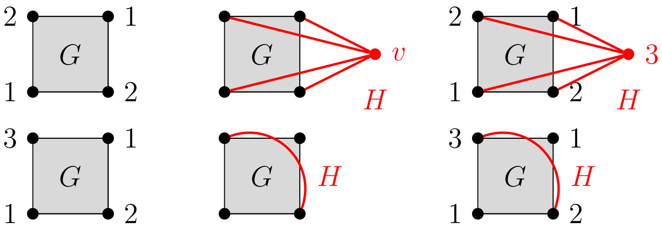

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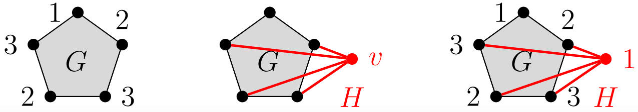


Some triangles?

Theorem (Grötzsch '59)

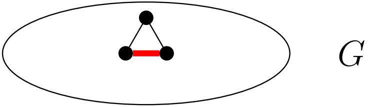
Every planar triangle-free graph is 3-colorable.

Some triangles?

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3*-colorable.*

We already showed one triangle!



Removing one edge of triangle results in triangle-free G.

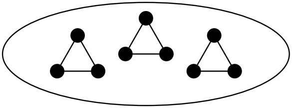
Some triangles?

Theorem (Grötzsch '59)

Every planar triangle-free graph is 3-colorable.

Theorem (Grünbaum '63; Aksenov '74; Borodin '97)

Let G be a planar graph containing at most three triangles. Then G is 3-colorable.



G

Three triangles - Proof outline

Theorem (Grünbaum '63; Aksenov '74; Borodin '97)
Let G be a planar graph containing at most three triangles. Then G is 3-colorable.

- G is 4-critical (minimal counterexample)
- 3-cycle is a face
- 4-cycle is a face or has a triangle inside and outside
- 5-cycle is a face or has a triangle inside and outside

CASE1: G has no 4-faces
CASE2: G has a 4-faces with triangle (no identification)
CASE3: G has a 4-face where identification applies

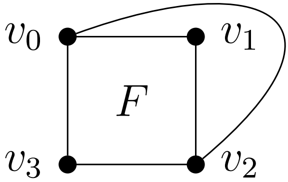
Three triangles - Proof outline

CASE1: G has no 4-faces

- $v - 2 + f = e$ by Euler's formula
- $5v - 4 + 5f - 6 = 5e$
- $2e \geq 5(f - 3) + 3 \cdot 3 = 5f - 6$ since 3 triangles
- $5v - 4 \geq 3e$ (our case)
- $3e \geq 5v - 2$ (every 4-critical graph)

Three triangles - Proof outline

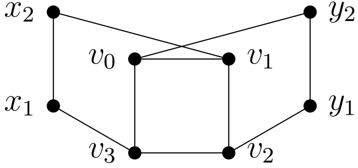
CASE2: G has a 4-face F with a triangle (no identification)



Both v_0, v_1, v_2 and v_0, v_2, v_3 are faces. G has 4 vertices!

Three triangles - Proof

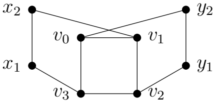
CASE3: G has a 4-face where identification applies



Since G is planar, some vertices are the same.

Three triangles - Proof

CASE3: G has a 4-face where identification applies



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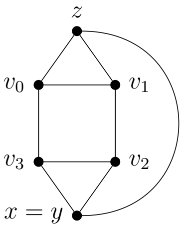

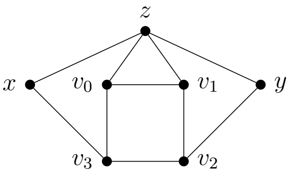

Three triangles - Proof

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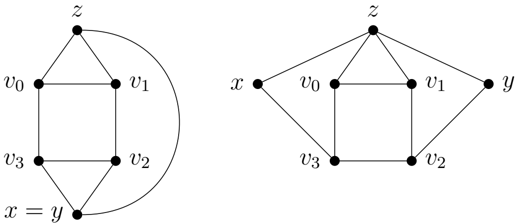
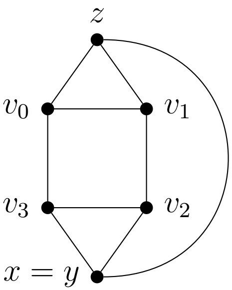


Only some cases left ...

One of the 2 cases left



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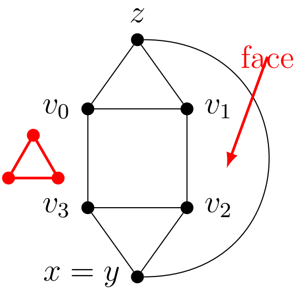

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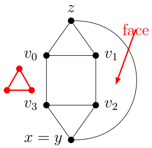

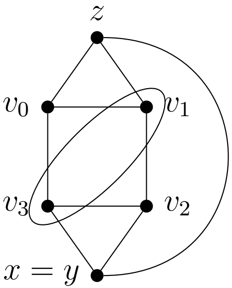

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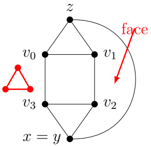
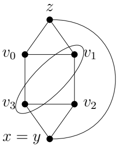
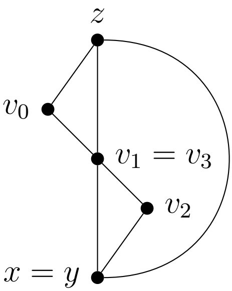
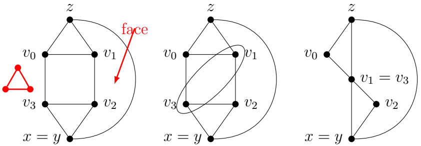
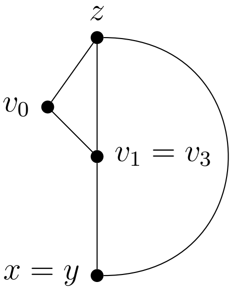






One of the 2 cases left





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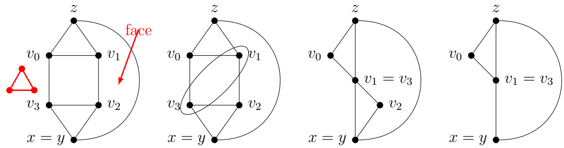

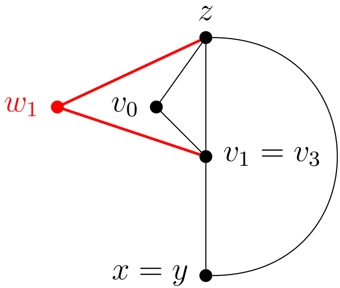

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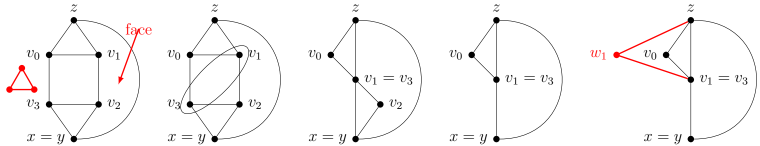

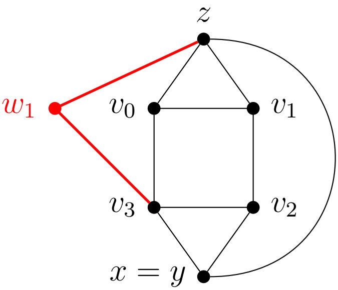

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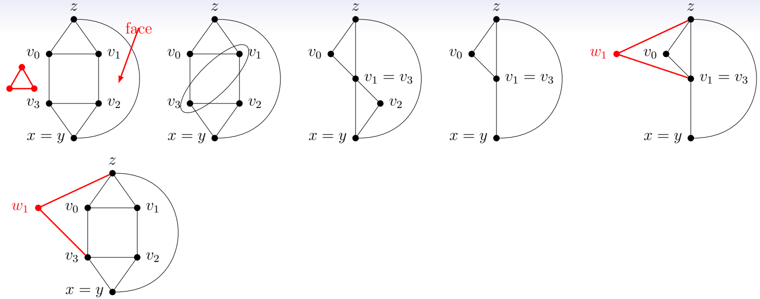

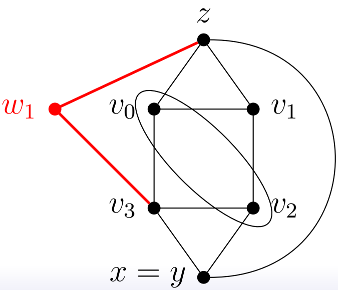

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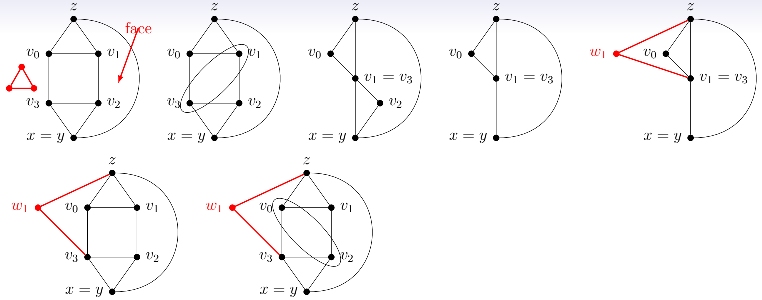

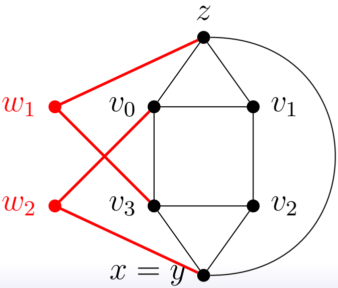

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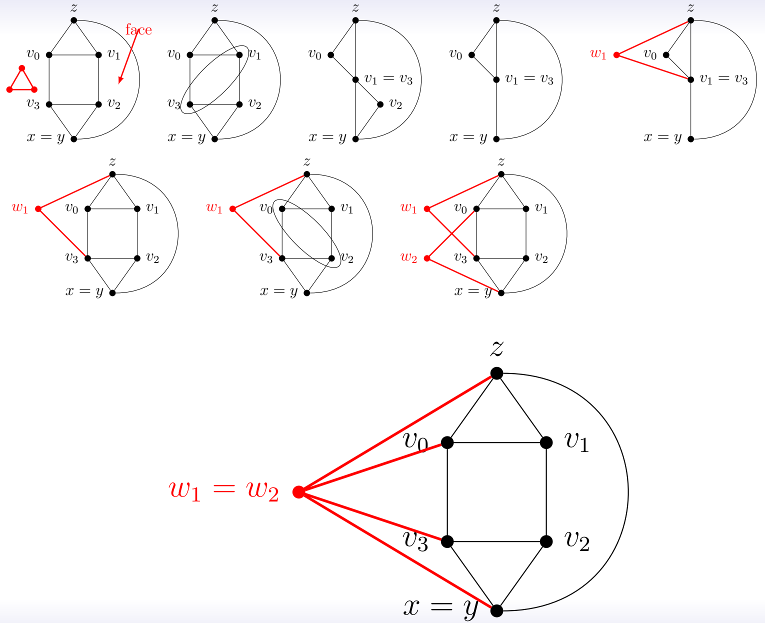

Thank you for your attention!

I hope to see you at EXCILL2, Mar 16-18 @ UIUC