The planar slope-number of planar partial 3-trees of bounded degree

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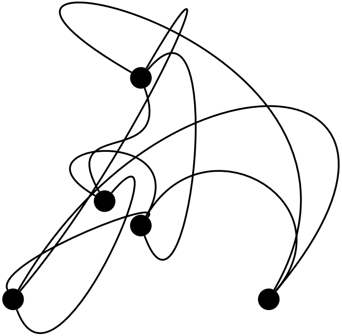
May 12, 2012

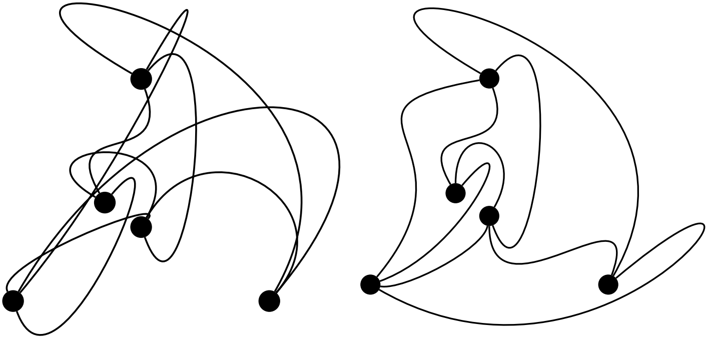
Graph drawing

graph $G = (V, E)$
Drawing maps V to points and E to curves.



Graph drawing

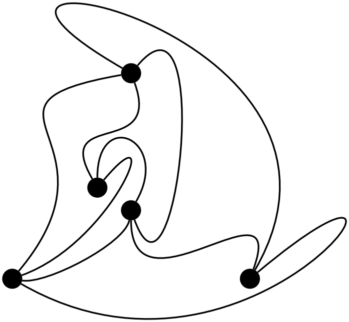
graph $G = (V, E)$
Drawing maps V to points and E to curves without crossings.

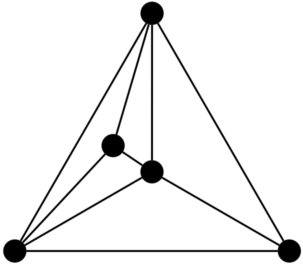


G is *planar* if it can be drawn without crossings

Graph drawing

graph $G = (V, E)$
Drawing maps V to points and E to straight segments.





straight-line drawing

Graph drawing

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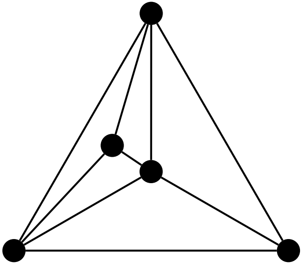
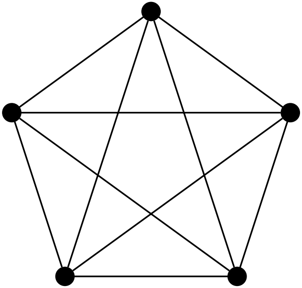




straight-line drawing does not have to be non-crossing

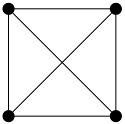
Definitions

Definition (Wade, Chu; 1994)

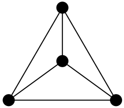
Slope-number sl(G) of a graph *G* is the minimum number of slopes in a straight-line drawing of *G*.

Definition (Dujmović, Suderman, Wood; 2004)

Planar-slope-number psl(G) of a planar graph *G* is the minimum number of slopes in a straight-line drawing of *G* without crossings.



$sl(K_4) = 4$



$psl(K_4) = 6$

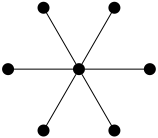
Recall that every planar graph has a straight-line drawing.

How to bound *sl*(*G*) and *psl*(*G*)?

Definition (Wade, Chu; 1994)

Slope-number sl(*G*) of a graph *G* is the minimum number of slopes in a straight-line drawing of *G*.

$sl(P_n) = 1$



$sl(S_n) = \lceil n/2 \rceil$

How to bound *sl*(G) and *psl*(G)?

Definition (Wade, Chu; 1994)

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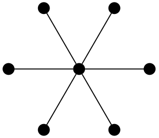

$$sl(P_n) = 1 \qquad sl(S_n) = \lceil n/2 \rceil$$

$$sl(P_n) = \lceil \Delta/2 \rceil \qquad sl(S_n) = \lceil \Delta/2 \rceil$$

where Δ is the maximum degree

$psl(G) \geq sl(G) \geq \lceil \Delta/2 \rceil$

Bounds on *sl*(G) (with crossings)

Theorem (Barát, Matoušek, Wood and indep. Pach, Pálvölgyi; 2006)

There exist graphs with $\Delta \geq 5$ *and arbitrarily large slope-number.*

Theorem (Keszegh, Pach, Pálvölgyi, Tóth; 2007)

If $\Delta \leq 3$ *then slope-number* ≤ 5*.*

Theorem (Mukkamala, Szegedy; 2009)

If $\Delta \leq 3$ *then slope-number* ≤ 4*.*

Case $\Delta = 4$ is not known.

Bounds on $psl(G)$ (no crossings)

Planar-slope-number $psl(G)$ of a planar graph G is the minimum number of slopes in a straight-line drawing of G without crossings.

Theorem (Dujmović, Epstein, Suderman, Wood 2007)

For every tree G holds $psl(G) = \lceil \Delta/2 \rceil$.
For cubic 3-connected planar grap G holds $psl(G) \leq 6$.
Bounds for maximal outerplanar graphs, planar 2-trees, planar 3-trees, planar 2-connected and 3-connected as a function of $|V(G)|$.

(outerplanar graph has a drawing with one face incident to all vertices)

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Problem (Dujmović, Epstein, Suderman, Wood 2007)

Is there a function f such that for every (outerplanar) graph G holds $psl(G) \leq f(\Delta(G))$.

(outerplanar graph has a drawing with one face incident to all vertices)

Answer to the problem

Theorem (JJKLTV; 2010)

Every partial planar 3-tree G *has* $psl(G) = O(c^{\Delta})$*.*
(constructive)

(outerplanar graphs $\subset$ partial planar 3-trees)

Theorem (Keszegh, Pach, Pálvölgyi; 2011)

Every planar graph G *has* $psl(G) = O(c^{\Delta})$ *for some constant* c*.*
(non-constructive)

Theorem (JJKLTV; 2012+)

Every partial planar 3-tree G *has* $psl(G) = O(\Delta^5)$*.*
(constructive)

Theorem (Knauer, Micek, Walczak 2012+)

Every outerplanar graph G *with* $\Delta \geq 4$ *has* $psl(G) \leq \Delta - 1$*.*
(constructive)

Planar 3-tree

A graph G is a 3-tree if it is

- a triangle
- obtained from a 3-tree by adding a vertex adjacent to three vertices of a triangle

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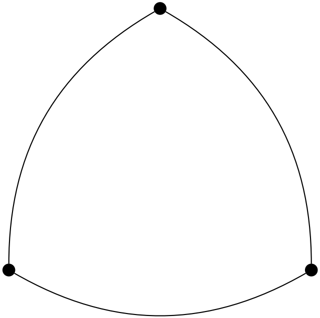

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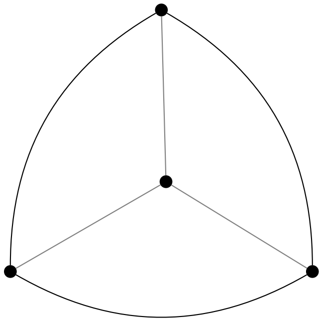

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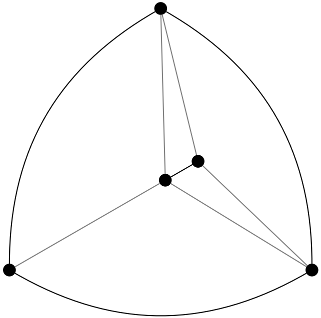

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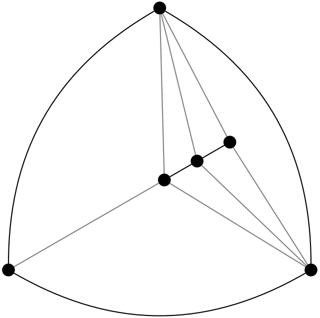

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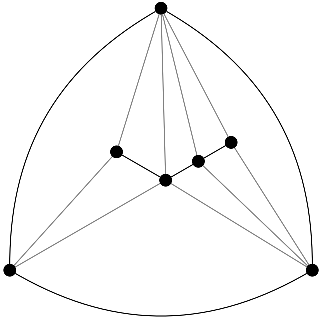

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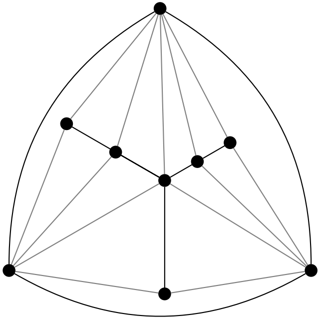

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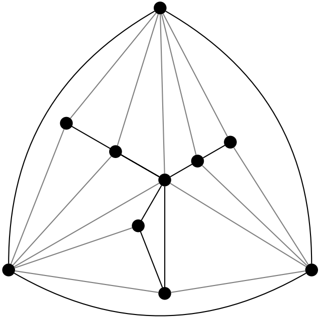

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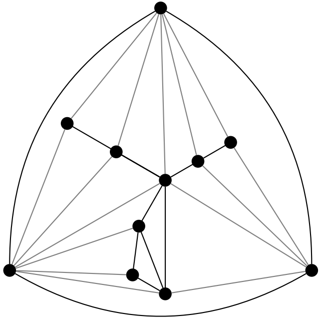

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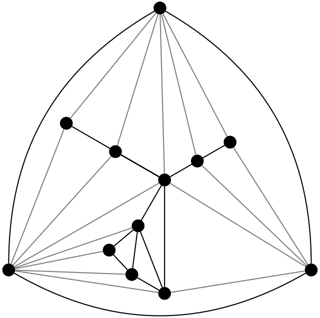

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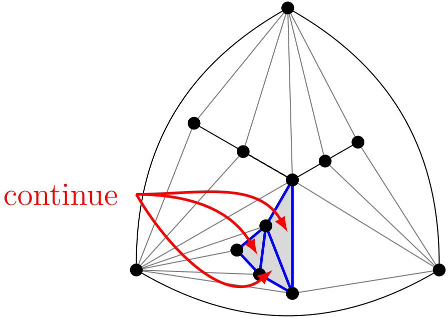

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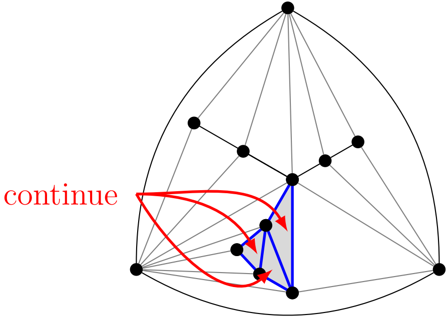

Partial 3-tree is a sugraph of a 3-tree

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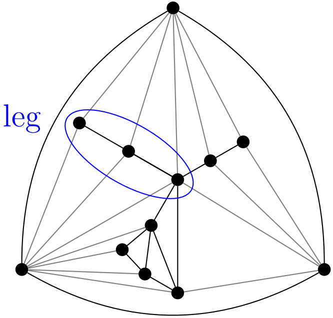

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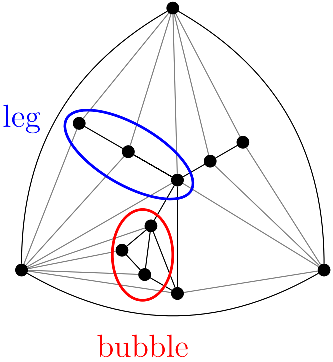
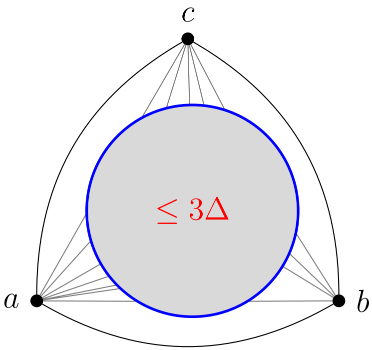


Partial 3-tree is a sugraph of a 3-tree

Proof idea - overview

Suppose every vertex is adjacent to a, b or c

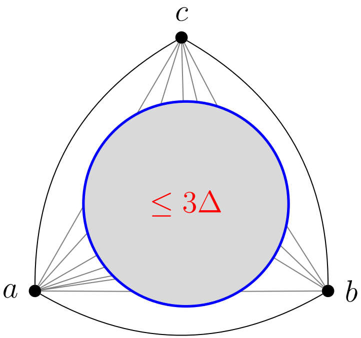
- $\leq 3\Delta$ vertices



Proof idea - overview

Suppose every vertex is adjacent to a, b or c

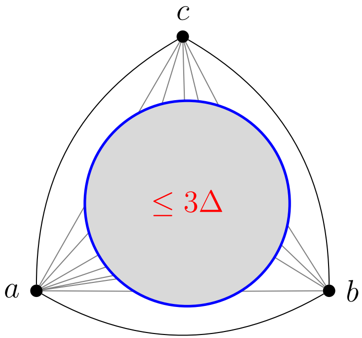
- $\leq 3\Delta$ vertices
- finitely many graphs (and hence drawings)



Proof idea - overview

Suppose every vertex is adjacent to a, b or c

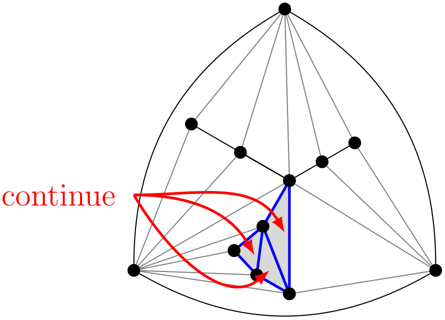
- $\leq 3\Delta$ vertices
- finitely many graphs (and hence drawings)
- finitely many slopes and triangles



Proof idea - overview

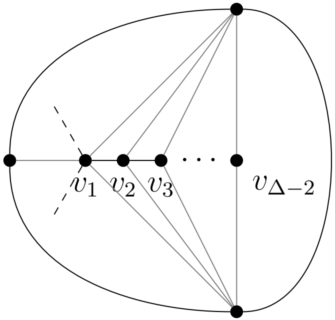
~~Suppose every vertex is adjacent to a, b or c~~

- $\leq 3\Delta$ vertices
- finitely many graphs (and hence drawings)
- finitely many slopes and triangles
- finitely many things to place in triangles (recursion)



Proof idea - drawing legs and bubbles

Draw leg on a line



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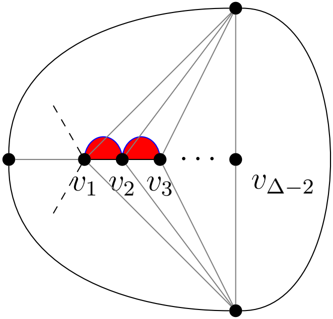
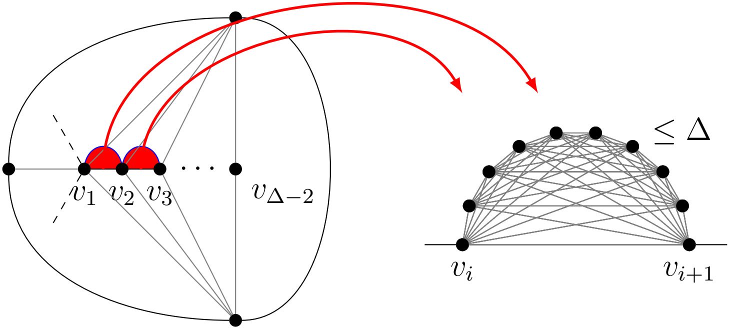


Proof idea - drawing legs and bubbles

Draw leg on a line and make all bubbles the same



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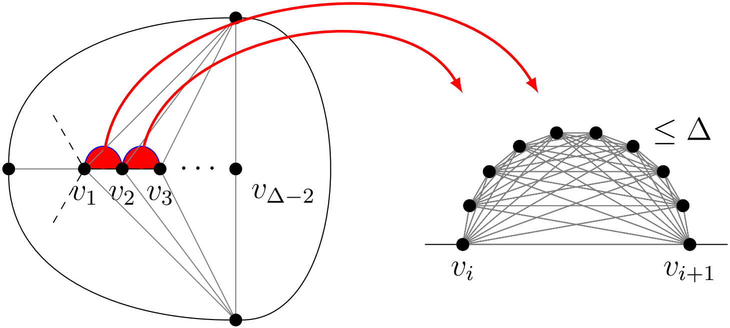
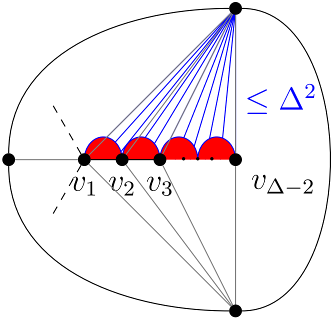
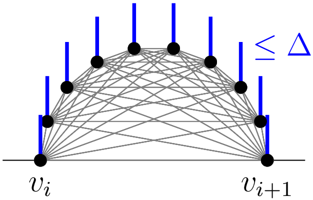


One leg and bubbles $O(\Delta^2)$ slopes and $O(\Delta^3)$ triangles.

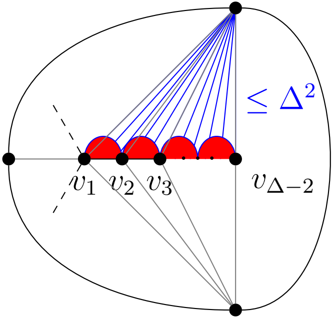
Proof idea - drawing legs and bubbles

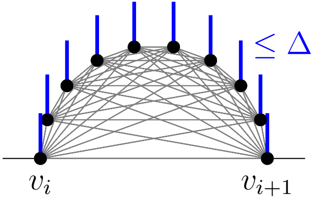




One leg and bubbles $O(\Delta^2)$ slopes and $O(\Delta^3)$ triangles.
One leg an bubbles have $O(\Delta^2)$ points, so $O(\Delta^2)$ slopes.

Proof idea - drawing legs and bubbles





One leg and bubbles $O(\Delta^2)$ slopes and $O(\Delta^3)$ triangles.
One leg an bubbles have $O(\Delta^2)$ points, so $O(\Delta^2)$ slopes.
Lines from placing legs and bubbles in triangles gives $O(\Delta^3 \cdot \Delta^2)$ slopes. Total $O(\Delta^5)$.

Open problems

Problem

Is slope-number graphs with $\Delta \leq 4$ bounded?

Problem

Is planar slope-number of planar graphs $O(\Delta^c)$?
(or constructive bound)

Problem

Is planar slope-number of planar graphs of bounded tree-width $O(\Delta^c)$?

Problem

Improve lower bound $3\Delta - 6$ for planar slope-number.

Thank you for your attention!